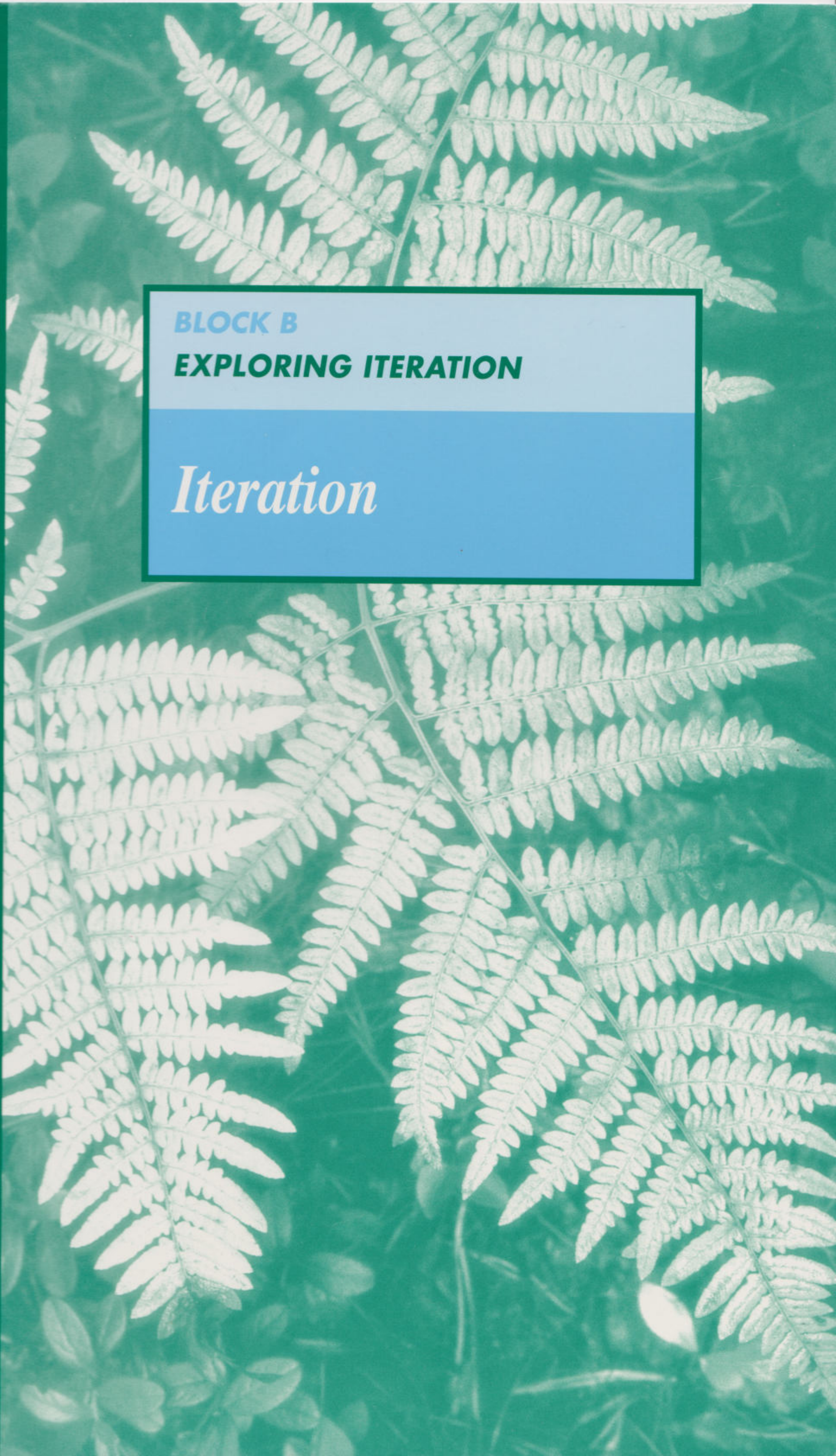
MS221 Chapter B1

The Open University

A second level interdisciplinary course



BLOCK B

EXPLORING ITERATION

Iteration

Exploring Mathematics

CHAPTER

B1

The Open University

A second level interdisciplinary course

Exploring Mathematics

CHAPTER

B1

BLOCK B

EXPLORING ITERATION

Iteration

Prepared by the course team

About this course

This course, MS221 *Exploring Mathematics*, and the courses MU120 *Open Mathematics* and MST121 *Using Mathematics* provide a flexible means of entry to university-level mathematics. Further details may be obtained from the address below.

MS221 uses the software program Mathcad (MathSoft, Inc.) to investigate mathematical concepts and as a tool in problem solving. This software is provided as part of the course.

This publication forms part of an Open University course. Details of this and other Open University courses can be obtained from the Student Registration and Enquiry Service, The Open University, PO Box 197, Milton Keynes, MK7 6BJ, United Kingdom: tel. +44 (0)870 333 4340, e-mail general-enquiries@open.ac.uk

Alternatively, you may visit the Open University website at http://www.open.ac.uk where you can learn more about the wide range of courses and packs offered at all levels by The Open University.

To purchase a selection of Open University course materials, visit the webshop at www.ouw.co.uk, or contact Open University Worldwide, Michael Young Building, Walton Hall, Milton Keynes, MK7 6AA, United Kingdom, for a brochure: tel. +44 (0)1908 858785, fax +44 (0)1908 858787, e-mail ouwenq@open.ac.uk

The Open University, Walton Hall, Milton Keynes, MK7 6AA.

First published 1997. Second edition 2002. Reprinted 2003. Reprinted with corrections 2005.

Edited, designed and typeset by The Open University, using the Open University TEX System.

Printed and bound in the United Kingdom by The Charlesworth Group, Wakefield.

ISBN 0 7492 3998 1

2.3

Contents

Introduction to Block B

The theme of Block B is *iteration*, which is the repeated application of a function, or process. For example, the generation of a first-order recurrence sequence, such as

See MST121 Chapter A1, Subsection 1.3.

$$x_0 = \tfrac{1}{2}, \quad x_{n+1} = x_n^2 \quad (n = 0, 1, 2, \ldots),$$

can be interpreted as iteration of a real function. Indeed, this sequence is obtained by repeated application of the real function $f(x) = x^2$, with initial term $\frac{1}{2}$, as indicated in Figure 0.1.

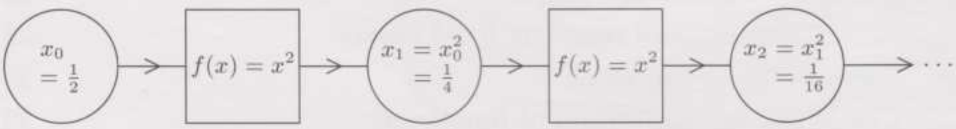


Figure 0.1 Iteration of $f(x) = x^2$

In Chapter B1, we study the long-term behaviour of *iteration sequences* of this type by using properties of the graph of the corresponding function f and by classifying the *fixed points* of f, those points that remain unchanged when the function f is applied.

In Chapter A3 you met a family of functions with domain and codomain $\mathbb{R}^2$, known as isometries. Isometries preserve distances between pairs of points (so the image of any set under an isometry is a congruent set). In Chapter B2, *linear transformations* are introduced. These have domain and codomain $\mathbb{R}^2$; they need not preserve distances between pairs of points, but they do map triangles to triangles (though not necessarily of the same shape and size).

In Chapter B3, we consider the effect of iterating a linear transformation f, and we find that the long-term behaviour of sequences of points in $\mathbb{R}^2$ produced by iteration of f can be understood by studying the lines which remain *invariant*, that is, unchanged, under f. In particular, results about iterating linear transformations can be used to establish properties of certain second-order population models.

Study guide

There are five sections to this chapter. They are intended to be studied consecutively in five study sessions. Each section will require two to three hours to study.

Section 2 requires the use of an audio cassette player, and Section 4 requires the use of the computer together with Computer Book B.

The pattern of study for each session might be as follows.

Study session 1: Section 1.
Study session 2: Section 2.
Study session 3: Section 3.
Study session 4: Section 4.
Study session 5: Section 5.

Section 5 is independent of the other sections, so it can be studied earlier if you wish.

Before studying this chapter, you should be familiar with the following topics:

- the idea of a convergent sequence and a sequence that tends to infinity;
- the equation of a line, and its gradient;
- the notation for open intervals and for closed intervals, for example (a, b), (a, ∞) and $[a, b]$;
- the notion of a real function (domain, rule, codomain) and its graph, and the notations for functions;
- sketching the graph of a quadratic function;
- basic properties of the modulus function $f(x) = |x|$, and interpretation and manipulation of simple expressions involving the modulus;
- the concepts of an increasing function and a decreasing function.

> The optional Video Band B(i) *Algebra workout – Binomial Theorem* could be viewed at any stage during your study of this chapter.

Introduction

This chapter deals with two types of iteration. The first of these is *iteration of real functions*, in which we study certain first-order recurrence sequences, here called *iteration sequences*. These sequences are generated by repeated application of a real function. In Section 1, you will meet a powerful graphical technique that makes it possible to understand the long-term behaviour of many such sequences.

This graphical technique is developed further in Section 2, using information about the *slope*, or *gradient*, of the graph of the function being iterated at points on that graph. In particular, conditions are given which guarantee that certain iteration sequences are convergent.

Section 3 introduces *composition of functions* – that is, producing a new function by applying one function to the outputs of another. This technique will be needed later in the course, so it is developed in appropriate generality. Here composition of functions is used to explain the rather strange long-term behaviour of certain iteration sequences which seem to tend to *two* different values!

In Section 4, you will use the computer to study properties of iteration sequences in much greater detail than is possible by hand calculation. You will discover that even iteration sequences which look simple can display a wide range of complicated long-term behaviour.

Finally, in Section 5 we study a rather different type of iteration, used to obtain the coefficients in the expansion of the *binomial* expression $(a+b)^n$. These coefficients can be derived by repeated application of a simple process which creates what is known as *Pascal's triangle*. You will see that the coefficients can also be calculated using a formula that is connected with a certain 'counting problem'.

In this chapter, the range of basic mathematical notation is extended to include some new symbols, which permit concise expression in mathematical arguments. To this end, the symbols $\in$ for 'in' or 'belongs to' (according to context), and $\subseteq$ for 'is a subset of' are introduced.

The symbols $\notin$ for 'does not belong to' and $\not\subseteq$ for 'is not a subset of' can also be used.

The modulus function is also used to achieve concise forms of expression. For example, the two inequalities $-1 < a < 1$ may be written as $|a| < 1$.

1 Iterating real functions

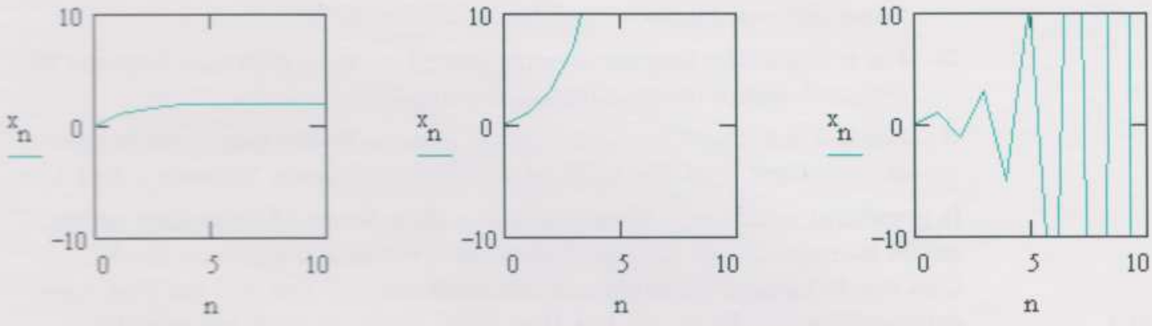
First-order recurrence sequences occur in many situations. For example, *linear recurrence sequences*, defined in the form

$$x_0 = a, \quad x_{n+1} = rx_n + d \quad (n = 0, 1, 2, \ldots), \tag{1.1}$$

A recurrence sequence is of first order if x_{n+1} depends only on x_n.

arise in calculating mortgage repayments and in population modelling. Such sequences have a closed form that expresses the general term x_n in terms of the subscript n and the parameters a, r and d. This closed form allows us to determine the long-term behaviour of linear recurrence sequences, some examples of which are shown in Figure 1.1 (in each case, $a = 0$ and $d = 1$).

See MST121 Chapter A1, Subsection 4.2.



(a) convergent (b) tends to infinity (c) unbounded

Figure 1.1 (a) $r = \frac{1}{2}$, (b) $r = 2$, (c) $r = -2$

- In Figure 1.1(a) the sequence x_n is **convergent**. This means that the terms **tend to** (that is, approach arbitrarily closely to) a number called the **limit** of the sequence; we write $x_n \to \ell$ as $n \to \infty$. We also say that a convergent sequence **converges** to its limit.
- In Figure 1.1(b) the sequence x_n **tends to infinity**. This means that the terms become arbitrarily large and positive for large values of n; we write $x_n \to \infty$ as $n \to \infty$.
- In Figure 1.1(c) the sequence x_n is **unbounded**. This means that the terms become arbitrarily large (not necessarily positive) for large values of n.

If the terms of a sequence x_n become arbitrarily large and *negative*, then x_n **tends to minus infinity** and we write

$$x_n \to -\infty \text{ as } n \to \infty.$$

Note that any sequence which tends to infinity or minus infinity is unbounded. Any sequence which is not convergent is said to be **divergent**. For example, any unbounded sequence is divergent.

Some first-order recurrence sequences display stranger types of long-term behaviour. For example, sequences defined by recurrence relations of the form

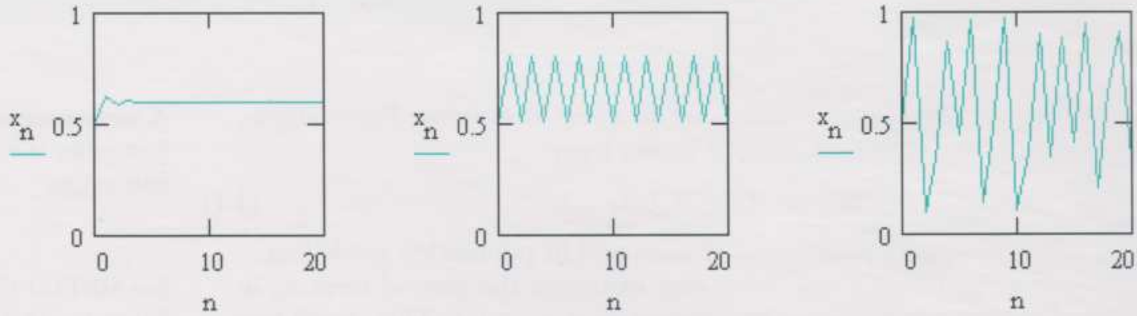
$$x_{n+1} = kx_n(1 - x_n) \quad (n = 0, 1, 2, \ldots) \tag{1.2}$$

arise in population modelling; here, k is a positive parameter, and the starting value is x_0. Such sequences have no closed form (except in special cases) and have a wide variety of possible long-term behaviours, illustrated in Figure 1.2, overleaf (in each case, $x_0 = 0.5$).

The recurrence relation (1.2) can be obtained from the version of the logistic recurrence relation given in MST121 Chapter B1,

$$x_{n+1} - x_n = rx_n(1 - x_n)$$
$$(n = 0, 1, 2, \ldots),$$

by substituting $x_n = \dfrac{k}{k-1} y_n$ and $r = k - 1$.



(a) convergent (b) alternating (c) no pattern

Figure 1.2 (a) $k = 2.5$, (b) $k = 3.2$, (c) $k = 3.9$

- ◇ For $k = 2.5$, the sequence x_n tends to a limit (approximately 0.6), the terms alternating above and below this limit.
- ◇ For $k = 3.2$, the long-term behaviour of x_n is to alternate between two *different* values (approximately 0.5 and 0.8).
- ◇ For $k = 3.9$, there is no discernible pattern to the long-term behaviour of x_n, except that the value of x_n seems to remain between 0 and 1.

It is natural to ask *why* these sequences show forms of behaviour which would be regarded as strange if they occurred in a population model. Can this behaviour be explained mathematically? You will see that some explanations *can* be given, but that a full explanation is not possible.

See Sections 3 and 4.

1.1 Graphical iteration

All the functions used in this chapter are real functions, so here is a reminder about the notation for such functions.

Real functions

The domain and codomain of a real function are both sets of real numbers.

In Chapter A3, the general definition of a function was introduced, involving a rule, domain and codomain. For example, the real function

$$\begin{aligned} f : [0,\infty) &\longrightarrow \mathbb{R} \\ x &\longmapsto \sqrt{x} \end{aligned}$$

has rule $f(x) = \sqrt{x}$ (or $x \longmapsto \sqrt{x}$), domain $[0, \infty)$ and codomain $\mathbb{R}$.

Real functions are often specified more concisely than in this two-line form. For example, the above function f can be specified as

$$f(x) = \sqrt{x} \quad (x \geq 0), \tag{1.3}$$

where the domain is indicated inside brackets. This function can be specified even more concisely, using the following convention.

This convention for the *domain* of a real function was introduced in MST121 Chapter A3, Section 1.

Domain and codomain convention

When the domain of a real function is not specified, it is understood to be the largest set of real numbers for which the rule is applicable.

When the codomain of a real function is not specified, it is understood to be $\mathbb{R}$.

According to this convention, the function in equation (1.3) can be specified just by

$$f(x) = \sqrt{x},$$

since $[0, \infty)$ is the largest set for which the rule of f is applicable, and the codomain is $\mathbb{R}$. Many of the functions used in this chapter can be specified in this way.

If we are interested only in positive values of x, then we might prefer to work with the function

$$g(x) = \sqrt{x} \quad (x > 0),$$

which has domain $(0, \infty)$ and codomain $\mathbb{R}$. Another common way to specify the domain of such a function is to use the symbol $\in$, read as 'in' or 'belongs to', as in the following specification of g:

$$g(x) = \sqrt{x} \quad (x \in (0, \infty)).$$

The symbol $\in$ is also useful in other situations, as you will see.

A graphical technique

To make any progress in understanding the long-term behaviour of general first-order recurrence sequences, several new techniques are needed. These techniques involve the real function associated with such a sequence. All first-order recurrence sequences are of the form

$$x_{n+1} = f(x_n) \quad (n = 0, 1, 2, \ldots), \tag{1.4}$$

where f is a given real function and x_0 is a given initial term. For example, in the case of a linear recurrence sequence (equation (1.1)), the function f is of the form $f(x) = rx + d$ and the initial term is $x_0 = a$. The recurrence relation in equation (1.4) can be visualised as repeated application of the function f, as shown in Figure 1.3.

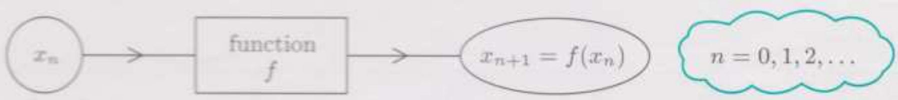


Figure 1.3 Repeated application of the function f

Since a first-order recurrence sequence is obtained by iterating a function f, it is often called an **iteration sequence**, and the sequence is said to be **generated** by f; also, the initial term of an iteration sequence is sometimes called the **starting value**.

Since all the functions considered in this chapter are real functions, the word 'real' is often omitted.

Activity 1.1 Identifying the function being iterated

For each of the following first-order recurrence relations, write down the function f being iterated.

(a) $x_{n+1} = x_n^2 + 1 \quad (n = 0, 1, 2, \ldots)$

(b) $x_{n+1} = \dfrac{1}{2}\left(x_n + \dfrac{3}{x_n}\right) \quad (n = 0, 1, 2, \ldots)$

Solutions are given on page 49.

For example, each recurrence sequence defined by equation (1.2) is generated by a quadratic function.

In this chapter, we mainly consider iteration sequences in which the function f is a quadratic function. This may seem to be only a small increase in complexity beyond linear recurrence sequences (equation (1.1)), but in terms of possible long-term behaviour, such sequences can be extremely complicated.

The next activity asks you to explore such iteration sequences in the case when $f(x) = x^2$, for several different initial terms.

Activity 1.2 Calculating iteration sequences

For each of the following initial terms x_0, calculate the first five terms of the iteration sequence given by

$$x_{n+1} = x_n^2 \quad (n = 0, 1, 2, \ldots).$$

Give your answers correct to three significant figures.

(a) $x_0 = 0$ (b) $x_0 = 1$ (c) $x_0 = 0.5$

(d) $x_0 = 1.2$ (e) $x_0 = 0.9$ (f) $x_0 = -0.9$

In each case, try to conjecture the sequence's long-term behaviour.

Solutions are given on page 49.

Comment

Note that if each initial term is replaced by its negative, then no new types of long-term behaviour are obtained (see parts (e) and (f)).

Activity 1.2 indicates that the long-term behaviour of a given iteration sequence may depend crucially on the value of the initial term. The behaviour of the iteration sequences in Activity 1.2 can be represented on the real line as shown in Figure 1.4.

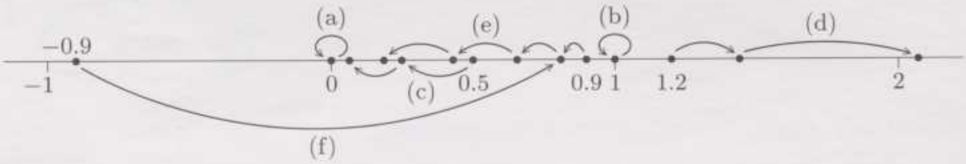


Figure 1.4 Iteration sequences on the real line

Each of the arrows represents an application of the function f, so Figure 1.4 gives a picture of the way in which iteration sequences generated by f behave.

As you will see, the line $y = x$ enables each output from f to become the next input for f.

A good way to *discover* how such an iteration sequence behaves, without calculating terms of the sequence, is to use **graphical iteration**. This is a geometric construction involving the graph of $y = f(x)$ and the line $y = x$, which is illustrated in Figure 1.5 in the case $f(x) = x^2$. Here graphical iteration is used to represent the sequence

$$x_0 = 0.9, \quad x_{n+1} = x_n^2 \quad (n = 0, 1, 2, \ldots),$$

whose terms you calculated in Activity 1.2(e).

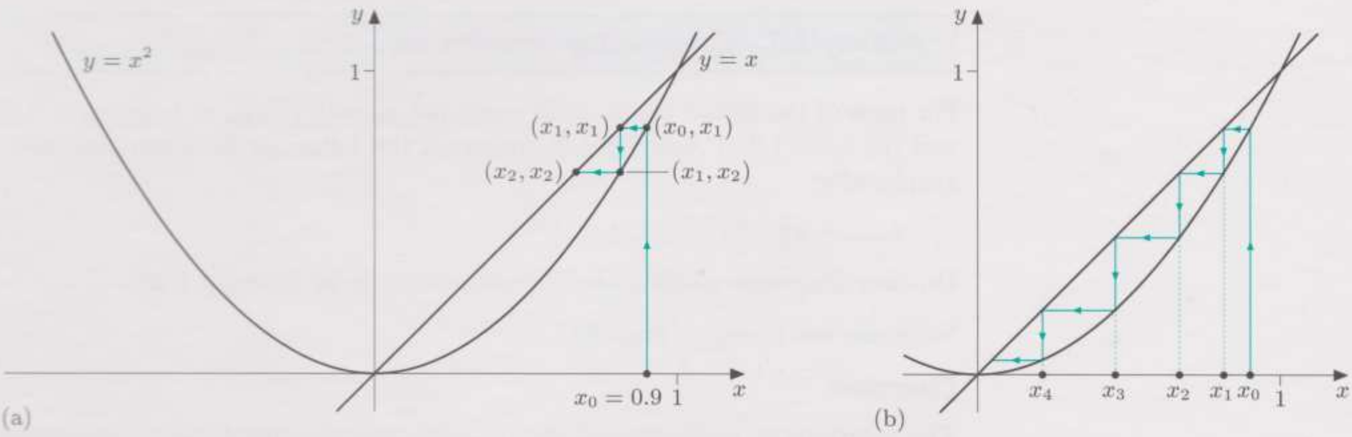


Figure 1.5 Graphical iteration with $f(x) = x^2$

Equal scales are used in these graphs.

To construct Figure 1.5(a), do the following.

The arrows show the direction of the construction.

- Start by sketching the graphs of $y = x$ and $y = x^2$, and marking $x_0 = 0.9$ on the x-axis.
- From there, draw a line vertically until you reach the graph of $y = f(x)$; the intersection occurs at the point

  $$(x_0, f(x_0)) = (x_0, x_1) = (0.9, (0.9)^2) = (0.9, 0.81).$$

- Then draw a horizontal line from (x_0, x_1) until you reach the line $y = x$; the intersection occurs at the point (x_1, x_1).

This horizontal line has equation $y = x_1$.

- Next, draw a line vertically from (x_1, x_1) until you reach the graph of $y = f(x)$; the intersection occurs at the point

  $$(x_1, f(x_1)) = (x_1, x_2) = (0.81, 0.656).$$

- Then draw a horizontal line from (x_1, x_2) until you reach the line $y = x$; the intersection occurs at the point (x_2, x_2).

Now repeat the last two steps, starting at the point (x_2, x_2), and then continue repeating the process

(1) draw a vertical line to meet $y = f(x)$,

(2) draw a horizontal line to meet $y = x$,

as long as the size and scale of the graph allow – see Figure 1.5(b). If each vertical line is extended to the x-axis, then this graphical construction allows you to find the position on the x-axis of the next term, $x_{n+1} = f(x_n)$, given the position of the current term, x_n.

In Figure 1.5(b), the construction lines form a 'staircase' leading down towards the origin, and the terms of the sequence x_n, shown on the x-axis, tend to 0. The construction indicates that

$$x_n \to 0 \text{ as } n \to \infty,$$

as conjectured in the solution to Activity 1.2(e).

This construction can be used quite generally to visualise sequences generated by iteration of a function f.

Note that, when using this construction, it is conventional to use the same scale for each axis.

Activity 1.3 Constructing iteration sequences

For each of the initial terms x_0 in parts (a) $x_0 = 0$, (b) $x_0 = 1$, (c) $x_0 = 0.5$ and (d) $x_0 = 1.2$ of Activity 1.2, construct the following iteration sequence graphically:

$$x_{n+1} = x_n^2 \quad (n = 0, 1, 2, \ldots).$$

Do your diagrams confirm the conjectures made in Activity 1.2?

Solutions are given on page 49.

Comment

The solutions to parts (a) and (b) indicate that this graphical construction sometimes terminates quickly.

In Activity 1.3 parts (c) and (d), the construction lines formed staircases; but sometimes a different shape emerges from this construction, as illustrated in Figure 1.6. Here, the graph of $y = f(x)$ slopes downhill (from left to right) as it crosses the line $y = x$. If this point of intersection has coordinates (a, a), then the result of the construction is a pattern of terms $x_0, x_1, x_2, \ldots$ lying alternately on either side of the point a on the x-axis.

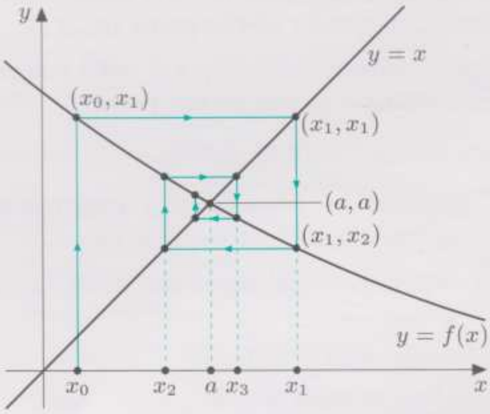


Figure 1.6 A cobweb

You have a chance to sketch such a 'cobweb' in the following activity.

Activity 1.4 Staircases and cobwebs

For each of the functions f and values of x_0 given in Figure 1.7, use the graph provided to construct the first few (at most four) terms of the iteration sequence generated by f, and describe the long-term behaviour of the iteration sequence.

Solutions are given on page 50.

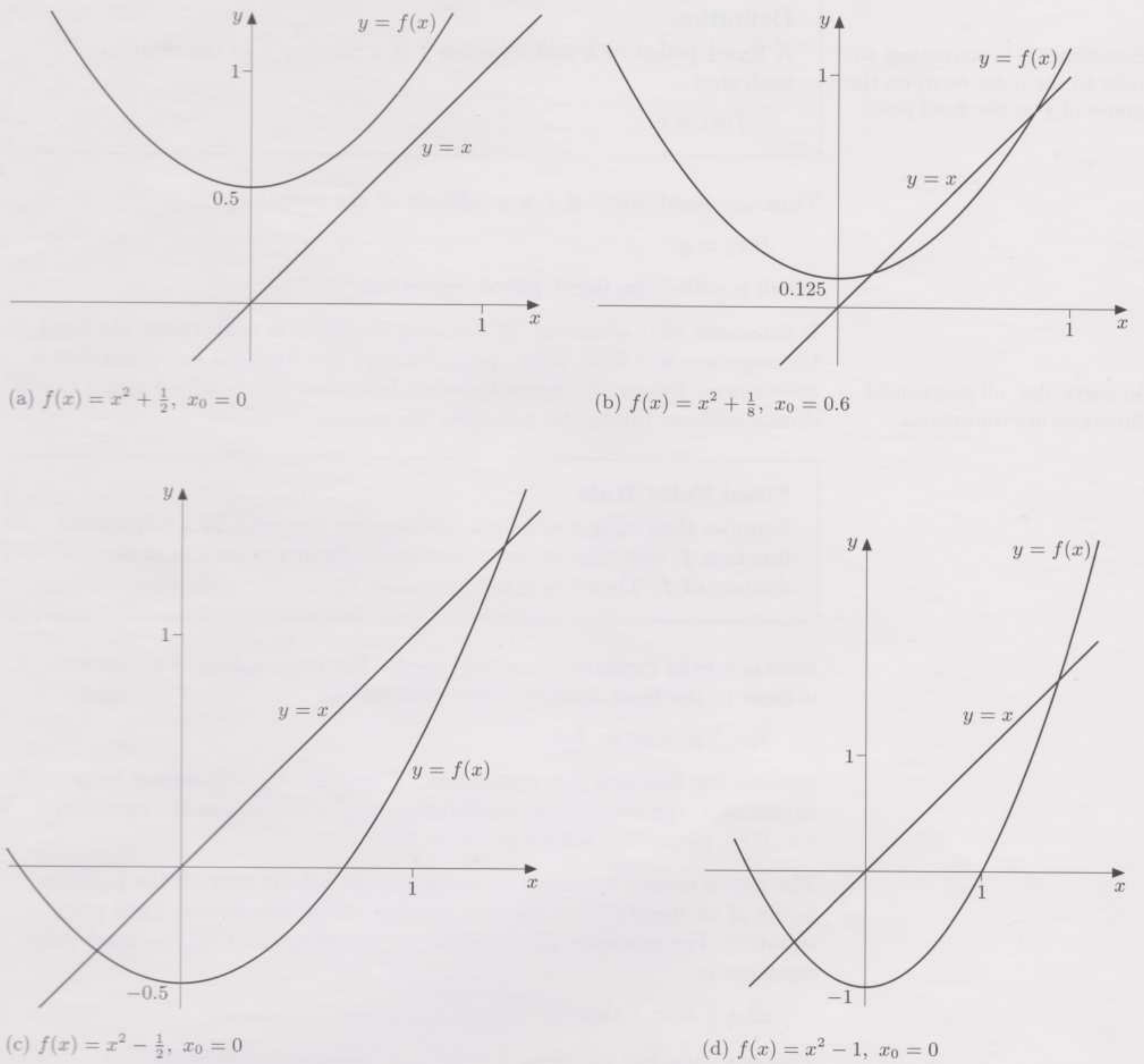


Figure 1.7 Various iteration sequences

1.2 Fixed points

In Activities 1.3 and 1.4, you saw several examples where an iteration sequence appears to converge to a limit. In each case when this occurs, the limit of the sequence is the x-coordinate of a point where the graph of $y = f(x)$ meets the line $y = x$. For example, in Activity 1.4(b), the sequence

$$x_0 = 0.6, \quad x_{n+1} = x_n^2 + \tfrac{1}{8} \quad (n = 0, 1, 2, \ldots), \tag{1.5}$$

appears to converge to the smaller of the two values of x for which $y = x^2 + \frac{1}{8}$ and $y = x$; that is, $x^2 + \frac{1}{8} = x$. Such a point is called a *fixed point* of the function.

Sometimes is is convenient to refer to the point (a, a) on the graph of f as the fixed point.

Definition

A **fixed point** of a real function f is a number a in the domain of f such that

$$f(a) = a.$$

Thus any fixed point of f is a solution of the equation

$$f(x) = x,$$

which is called the **fixed point equation**.

In particular, all polynomial functions are continuous.

It turns out that whenever an iteration sequence is convergent, the limit of the sequence is a fixed point, provided that the function being iterated is *continuous*. Informally, a **continuous function** is one whose graph can be drawn without lifting the pen from the paper.

Fixed Point Rule

Suppose that x_n is a sequence obtained by iteration of a continuous function f, and that x_n converges to the limit ℓ, which is in the domain of f. Then ℓ is a fixed point of f.

Here is a brief explanation of this result. For large values of n, the term x_n is close to the limit ℓ; so, for such values of n,

$$f(x_n) \text{ is close to } f(\ell),$$

because the function f is continuous. Therefore, by considering large values of n, the recurrence relation $x_{n+1} = f(x_n)$ leads to the equation $\ell = f(\ell)$. Hence ℓ is a fixed point of f.

The useful consequence of this result is that we can find all the possible limits of an iteration sequence by solving the corresponding fixed point equation. For example, for the sequence in equation (1.5), the fixed point equation is

$$x^2 + \tfrac{1}{8} = x; \quad \text{that is,} \quad x^2 - x + \tfrac{1}{8} = 0.$$

This equation has solutions $\frac{1}{2} \pm \frac{1}{4}\sqrt{2}$, so the graph of the function $f(x) = x^2 + \frac{1}{8}$ meets the line $y = x$ when $x = \frac{1}{2} + \frac{1}{4}\sqrt{2}$ and when $x = \frac{1}{2} - \frac{1}{4}\sqrt{2}$. In fact, the sequence defined by equation (1.5) converges to the smaller of these two fixed points, namely to

$$\tfrac{1}{2} - \tfrac{1}{4}\sqrt{2} \simeq 0.146,$$

as suggested by Activity 1.4(b).

Activity 1.5 Finding the limit

As you saw in Activity 1.4(c), the sequence

$$x_0 = 0, \quad x_{n+1} = x_n^2 - \tfrac{1}{2} \quad (n = 0, 1, 2, \ldots)$$

generated by the function $f(x) = x^2 - \frac{1}{2}$ appears to converge to a limit between -0.5 and 0. Assuming that this convergence does occur, find the value of the limit.

A solution is given on page 51.

So far in this section, most of the iteration sequences have been generated by functions of the form $f(x) = x^2 + c$. In the final activity for this section (Activity 1.6), you are asked to determine the long-term behaviour of an iteration sequence generated by a quadratic function of more general form. The graph of the quadratic function used in this activity is sketched in Example 1.1. Since the function is quadratic, its graph is a parabola.

Example 1.1 Sketching a quadratic graph

Sketch the graph of the function

$$f(x) = -\tfrac{1}{8}x^2 + \tfrac{11}{8}x + \tfrac{1}{2}.$$

Solution

First we express $f(x)$ in completed-square form. We have

$$\begin{aligned} f(x) &= -\tfrac{1}{8}(x^2 - 11x) + \tfrac{1}{2} \\ &= -\tfrac{1}{8}\left(x - \tfrac{11}{2}\right)^2 + \tfrac{1}{8}\left(\tfrac{11}{2}\right)^2 + \tfrac{1}{2} \\ &= -\tfrac{1}{8}\left(x - \tfrac{11}{2}\right)^2 + \tfrac{137}{32}. \end{aligned}$$

Hence the graph of f is a parabola with vertex $\left(\tfrac{11}{2}, \tfrac{137}{32}\right)$, and this is the highest point of the graph. (The graph of f can be obtained from the graph of $y = x^2$ by performing first a y-scaling with factor $-\tfrac{1}{8}$, and then a horizontal translation by $\tfrac{11}{2}$ units to the right and a vertical translation by $\tfrac{137}{32}$ ($\simeq 4.28$) units upwards.)

Sketching the graph of the function

$$f(x) = a(x + p)^2 + q$$

is discussed in detail in MST121 Chapter A3, Section 2. In particular, from Frame 11 of that chapter, the vertex is at $(-p, q)$, the axis of symmetry is $x = -p$, and the vertex is the

$$\begin{cases} \text{lowest point} & \text{if } a > 0, \\ \text{highest point} & \text{if } a < 0. \end{cases}$$

The y-intercept is $f(0) = \tfrac{1}{2}$, and the x-intercepts are the solutions of the equation $f(x) = 0$; that is,

$$-\tfrac{1}{8}x^2 + \tfrac{11}{8}x + \tfrac{1}{2} = 0.$$

The solutions are

$$x = \tfrac{1}{2}(11 \pm \sqrt{137}).$$

(Note that these solutions can be found from the quadratic formula, or by using the above completed-square form.) Thus the x-intercepts are approximately

$$-0.35 \quad \text{and} \quad 11.35.$$

The graph of f is shown in Figure 1.8.

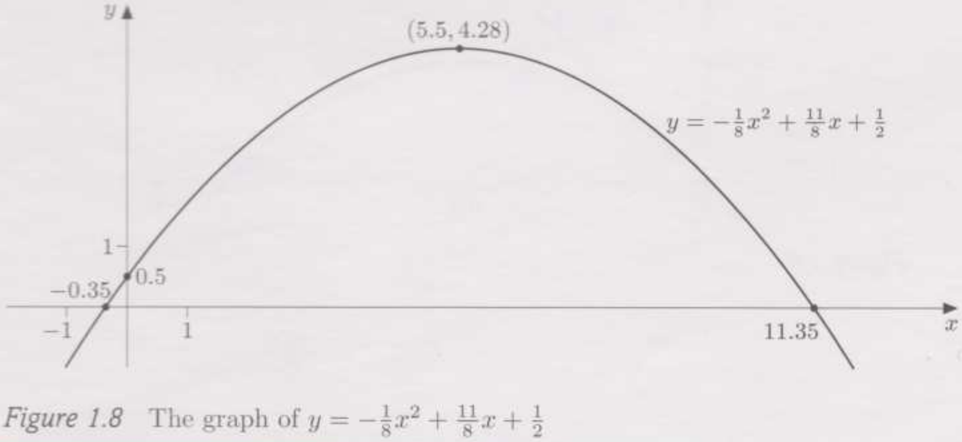


Figure 1.8 The graph of $y = -\tfrac{1}{8}x^2 + \tfrac{11}{8}x + \tfrac{1}{2}$

You should make use of Figure 1.8 in part (b) of the following activity.

Activity 1.6 Discovering long-term behaviour

(a) Determine the fixed points of the function $f(x) = -\frac{1}{8}x^2 + \frac{11}{8}x + \frac{1}{2}$.

(b) Use graphical iteration to determine the long-term behaviour of iteration sequences given by

$$x_{n+1} = -\tfrac{1}{8}x_n^2 + \tfrac{11}{8}x_n + \tfrac{1}{2} \quad (n = 0, 1, 2, \ldots)$$

for each of the following initial terms.

(i) $x_0 = 0$ (ii) $x_0 = -2$ (iii) $x_0 = 5$

Solutions are given on page 51.

Summary of Section 1

This section has introduced:

- generation of sequences by iterating functions;
- the process of graphical iteration for visualising iteration sequences;
- the concept of a fixed point of a function;
- the Fixed Point Rule.

Exercises for Section 1

Exercise 1.1

For each of the following iteration sequences, calculate the first four terms of the sequence (correct to three significant figures), and construct these terms, where possible, using the graph provided.

(a) $x_0 = 0.5, \quad x_{n+1} = x_n(1 - x_n) \quad (n = 0, 1, 2, \ldots)$

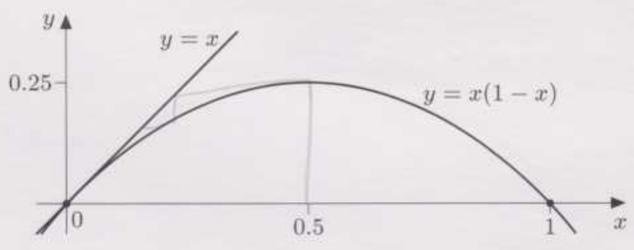


Figure 1.9

(b) $x_0 = 1, \quad x_{n+1} = \frac{1}{2}(x_n + 3/x_n) \quad (n = 0, 1, 2, \ldots)$

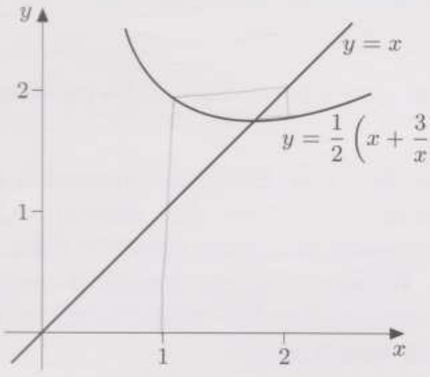


Figure 1.10

(c) $x_0 = 0, \quad x_{n+1} = \cos(x_n) \quad (n = 0, 1, 2, \ldots)$

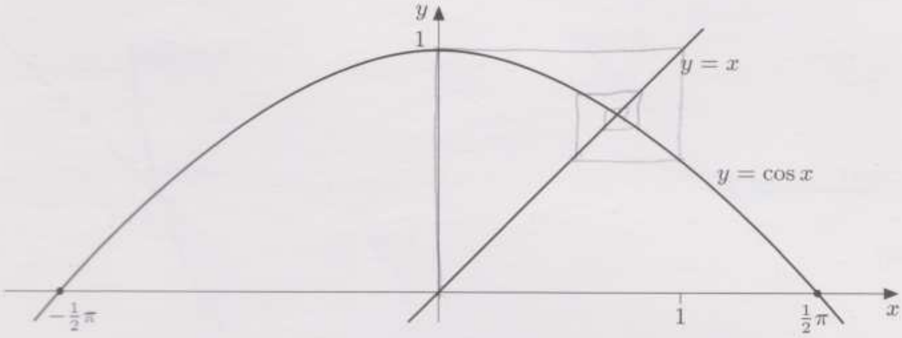


Figure 1.11

(d) $x_0 = 0, \quad x_{n+1} = x_n^2 - 2.4 \quad (n = 0, 1, 2, \ldots)$

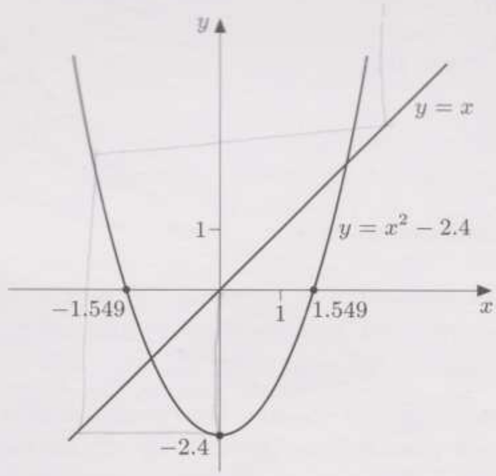


Figure 1.12

Exercise 1.2

Given that the iteration sequences in Exercise 1.1(a) and (b) are both convergent, find the limit of each of these sequences.

Exercise 1.3

(a) Given that the iteration sequence in Exercise 1.1(c) is convergent, what can you say about the value of the limit of this sequence?

(b) Describe the long-term behaviour of the sequence in Exercise 1.1(d).

2 Classifying fixed points

To study this section, you will need an audio cassette player and Audio Tape 1.

In Section 1 you saw that if an iteration sequence is convergent, then its limit is a fixed point of the function that generates the sequence. Next, we consider *which* fixed points of a given function f are limits of convergent iteration sequences. We again use the graphical construction of iteration sequences, and we examine how the 'shape' of the graph of $y = f(x)$ near a fixed point affects the construction.

Figure 2.1 shows graphical iteration in four cases, resulting in staircases ((a) and (b)) and cobwebs ((c) and (d)), some convergent and some divergent.

For clarity, the axes have been omitted.

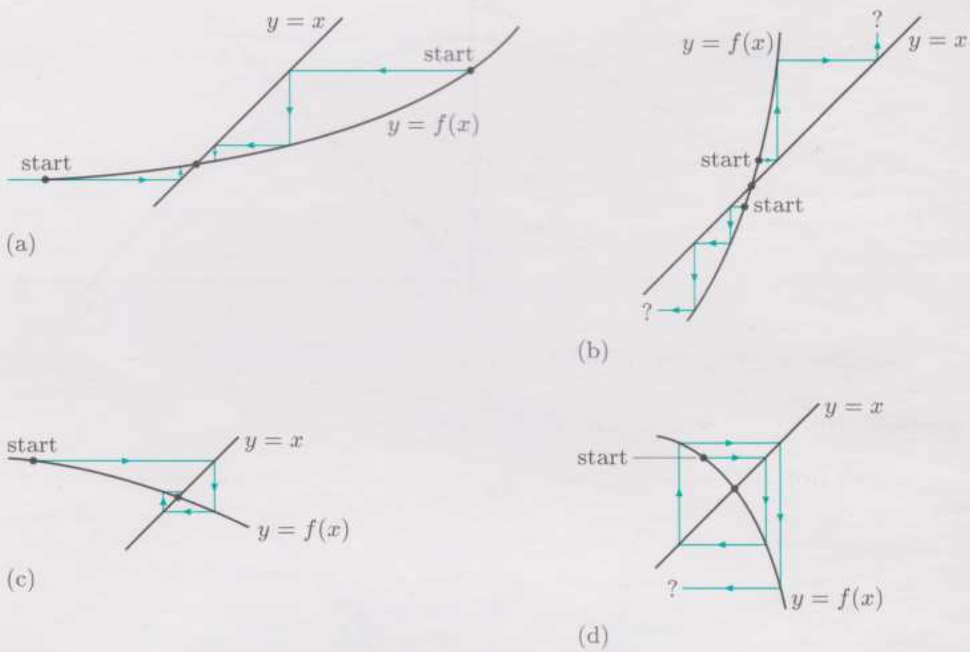


Figure 2.1 Staircases and cobwebs

The three iteration sequences in Figure 2.1(a) and (c) converge to the fixed point shown, whereas those in Figure 2.1(b) and (d) move away from it. What makes (a) and (c) different from (b) and (d)?

A major difference is that in (a) and (c) the graph of $y = f(x)$ is less steep near the fixed point than it is in (b) and (d). To be more precise about this difference, we need to have a measure of the 'steepness' of a graph, and this measure is now discussed.

2.1 The gradient of a graph

For a linear function $f(x) = mx + c$, we measure the steepness of the graph of $y = f(x)$ by using the *gradient*, or *slope*, m; for example, the gradient of the graph of $y = x$ is 1. The graph of a linear function is equally steep at all points, but this is not true for graphs of general functions. In general, we need to talk about the **gradient**, or slope, of a graph *at a point* on the graph. This is the gradient of the unique line that 'touches' the graph at the point, as in Figure 2.2. This line, the one which most closely approximates the graph of $y = f(x)$ near the point, is called the **tangent** to the graph at the given point on the graph.

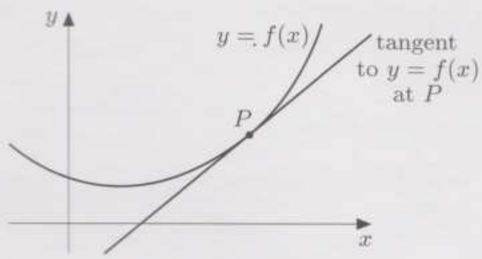


Figure 2.2 Gradient at a point on a graph

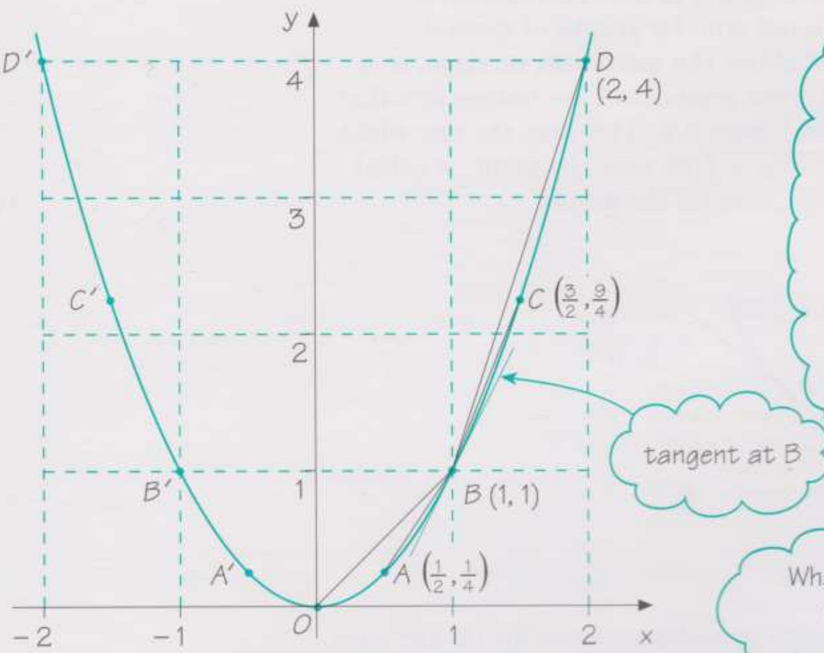
In Block C you will meet *calculus*, which provides methods for finding such gradients for a wide variety of functions. You will see there that, for many functions f, the graph of f has a tangent at *every* point; moreover, the gradients of such tangents can often be expressed by a formula related to the rule of f. Informally, a function is said to be **smooth** if there is a tangent at each point of its graph. For example, all polynomial functions are smooth.

In this section, we consider the iteration of *quadratic* functions, so we need to know the gradients of quadratic graphs. These are discussed in the audio tape that follows.

Now listen to Audio Tape 1, Band 3, 'Gradients of quadratic graphs'.

Frame 1

The graph of $y = x^2$



Chord line segment joining two points on a curve:

$$\text{gradient} = \frac{\text{rise}}{\text{run}}.$$

For BD,

$$\frac{\text{rise}}{\text{run}} = \frac{4-1}{2-1} = 3.$$

What is the gradient of the tangent at B?

Chord	OB	AB	BC	BD
Gradient	1	3/2	5/2	3

Frame 2

Gradient of tangent at (1, 1)

Gradient of chord with endpoints

$(1,1)$ and $(1 + h, (1 + h)^2)$ is

$$\frac{\text{rise}}{\text{run}} = \frac{(1+h)^2 - 1}{h}$$

$$= \frac{1 + 2h + h^2 - 1}{h}$$

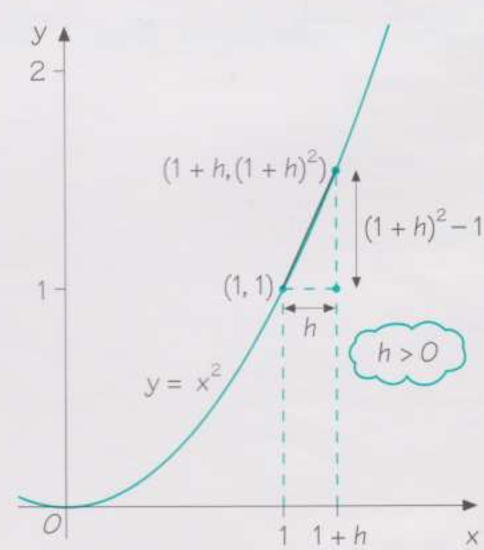
$$= \frac{2h + h^2}{h} = 2 + h.$$

When h is small:

◊ gradient of chord is close to gradient of tangent at (1,1);

◊ $2 + h$ is close to 2.

So gradient of tangent at (1,1) is 2.

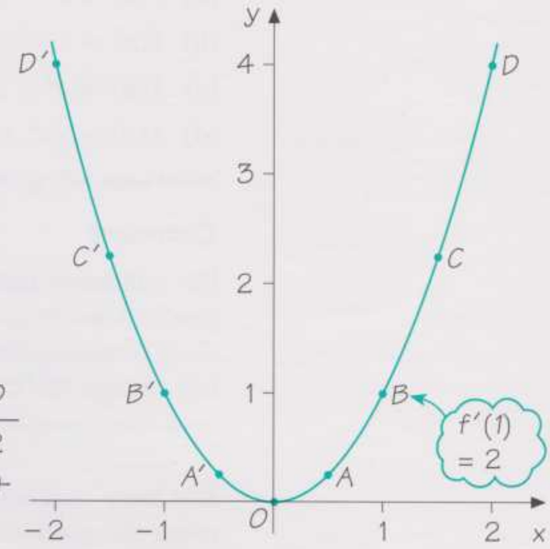


Frame 3

Gradient formula for $f(x) = x^2$

Gradient of $y = f(x)$ at $(x, f(x))$:
$f'(x) = 2x$.

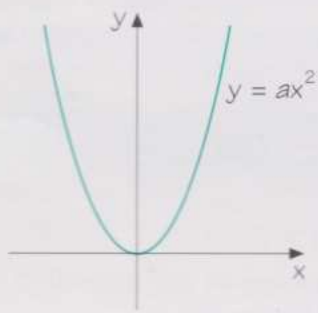
gradient of tangent at $(x, f(x))$



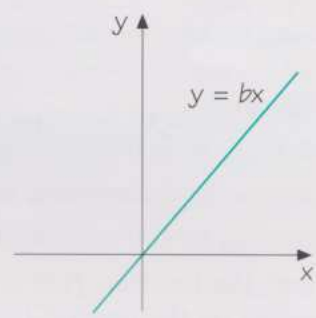
Point	D′	C′	B′	A′	O	A	B	C	D
x	-2	$-1\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2
$f'(x) = 2x$	-4	-3	-2	-1	0	1	2	3	4

Frame 4

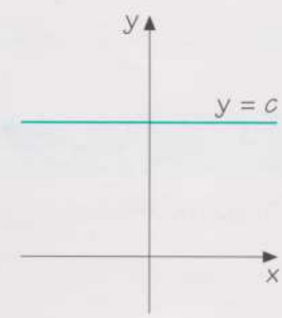
Gradient formula for $f(x) = ax^2 + bx + c$



gradient: $2ax$



gradient: b



gradient: 0

Gradient of $y = f(x)$ at $(x, f(x))$:
$f'(x) = 2ax + b$.

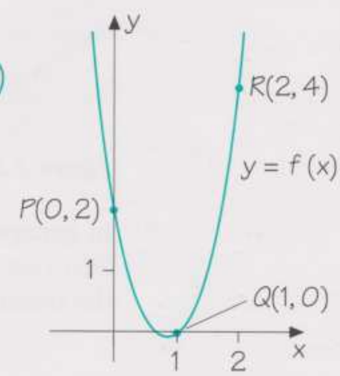
General quadratic gradient formula

Example

Let $f(x) = 3x^2 - 5x + 2$.

Find the gradient of $y = f(x)$ at P, Q and R.

$a = 3$, $b = -5$, $c = 2$



$f'(x) = 6x - 5$

Gradient at P: $6 \times 0 - 5 = -5$

Gradient at Q: $6 \times 1 - 5 = 1$

Gradient at R: $6 \times 2 - 5 = 7$

Activity 2.1 Finding gradients

Use the quadratic gradient formula in Frame 4 to find the gradient at the point given on the graph of $y = f(x)$ in each of the following cases.

(a) $f(x) = x^2$, $(\frac{1}{2}, \frac{1}{4})$

(b) $f(x) = -x^2 + 1$, $(3, -8)$

(c) $f(x) = x^2 - 2$, $(-\sqrt{2}, 0)$

(d) $f(x) = \frac{1}{3}x^2 + 7x + 1$, $(2, \frac{49}{3})$

Solutions are given on page 51.

Comment

For quadratic functions of the form $f(x) = x^2 + c$, the formula for the gradient is always the same, namely $f'(x) = 2x$. This is to be expected, since changing the constant c translates the graph vertically, which does not change the gradient of the graph at points with a given x-coordinate.

The concepts of 'increasing' and 'decreasing' were introduced in MST121 Chapter A3, Section 4.

You have now seen how to find the gradient of a quadratic graph. Before using the gradient to classify fixed points, it is useful to introduce another concept, which is closely related to the gradient. A real function f is said to be **increasing** if it has the property that

for all x_1, x_2 in the domain of f, if $x_1 < x_2$, then $f(x_1) < f(x_2)$.

The concept of a **decreasing** function is defined in a similar way:

for all x_1, x_2 in the domain of f, if $x_1 < x_2$, then $f(x_1) > f(x_2)$.

For many purposes, a more flexible concept is needed, since a function can be increasing on one interval and decreasing on another. Suppose that f is a real function whose domain contains an interval I. Then f is said to be **increasing on** I if

for all x_1, x_2 in I, if $x_1 < x_2$, then $f(x_1) < f(x_2)$,

and f is said to be **decreasing on** I if

for all x_1, x_2 in I, if $x_1 < x_2$, then $f(x_1) > f(x_2)$.

For example, the function $f(x) = x^2$ is increasing on the interval $[0, \infty)$, and decreasing on the interval $(-\infty, 0]$; see Figure 2.3.

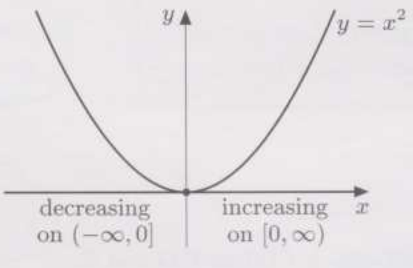


Figure 2.3 Graph of $y = x^2$

In general, if f is a quadratic function, then the real line can be divided into two intervals, with f increasing on one interval and f decreasing on the other. For other functions the situation can be more complicated.

As you will see in Block C, for a given smooth function f, the gradient can be used to determine on which intervals a function is increasing and on which it is decreasing. In this chapter, however, we always determine such intervals from the graph of the function.

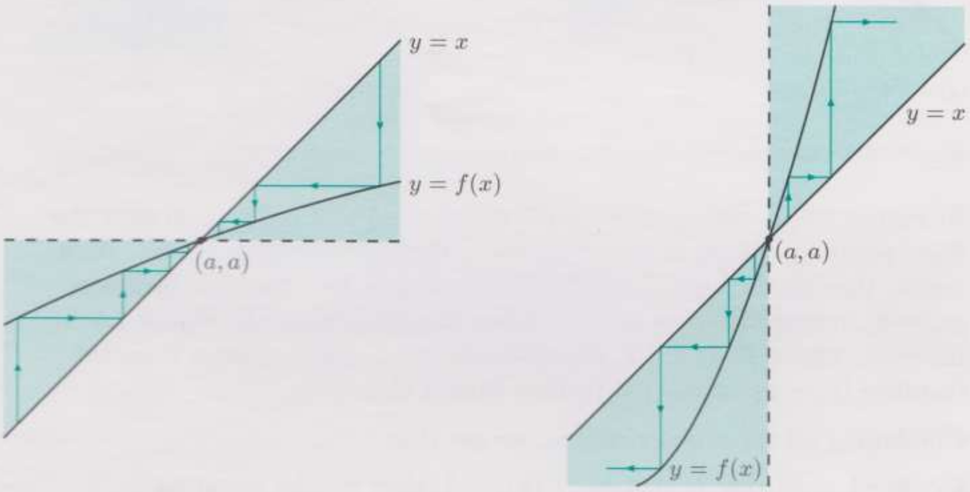
For example, the sine function is increasing on each of the intervals $\ldots, [-\frac{1}{2}\pi, \frac{1}{2}\pi], [\frac{3}{2}\pi, \frac{5}{2}\pi], \ldots$.

2.2 Attracting and repelling fixed points

Having introduced the idea of the gradient of the graph of $y = f(x)$ at a point on the graph, we now return to the problem of distinguishing between those iteration sequences which converge to a fixed point and those which do not; see Figure 2.1.

As suggested earlier, it is the gradient of the graph at the fixed point which largely determines how nearby iteration sequences behave. To see why, we consider an iteration sequence x_n generated by a smooth function f with fixed point a. Figure 2.4 shows several such sequences in the case when $f'(a) > 0$, where the graph of $y = f(x)$ slopes uphill (from left to right) near a.

A function is smooth if there is a tangent at each point of its graph.



(a) $0 < f'(a) < 1$ (b) $f'(a) > 1$

Figure 2.4 Graphs with positive gradient at a

In Figure 2.4(a) the gradient $f'(a)$ is less than 1, the gradient of the line $y = x$, so the graph of $y = f(x)$ is less steep than 45° near the point (a, a), and hence nearby iteration sequences tend to a. In Figure 2.4(b) the gradient $f'(a)$ is greater than 1, so the graph of $y = f(x)$ is steeper than 45° near the point (a, a), and hence nearby iteration sequences move away from a.

Since we use the same scale on each axis in graphical iteration, the line $y = x$ is at 45° to the x-axis.

The position is less clear in the case when the gradient $f'(a)$ is negative, so the graph of $y = f(x)$ slopes downhill (from left to right) near a. The diagrams in Figure 2.5, overleaf, show how the size of the gradient $f'(a)$ affects the distance from a of successive terms of a nearby iteration sequence x_n.

In Figure 2.5, the distance between points has been expressed in terms of the modulus function. The distance from one point p to another point q on the number line, which is a non-negative quantity, is given by

In particular, $|p|$ is the distance from p to 0.

$$|p-q| = \begin{cases} p-q, & \text{if } p \geq q, \\ q-p, & \text{if } p < q. \end{cases}$$

This representation is equally convenient for any vertical or horizontal line in the plane. For example, in Figure 2.5(b), the distance from (x_n, x_{n+1}) to (x_n, a) is $|x_{n+1} - a|$.

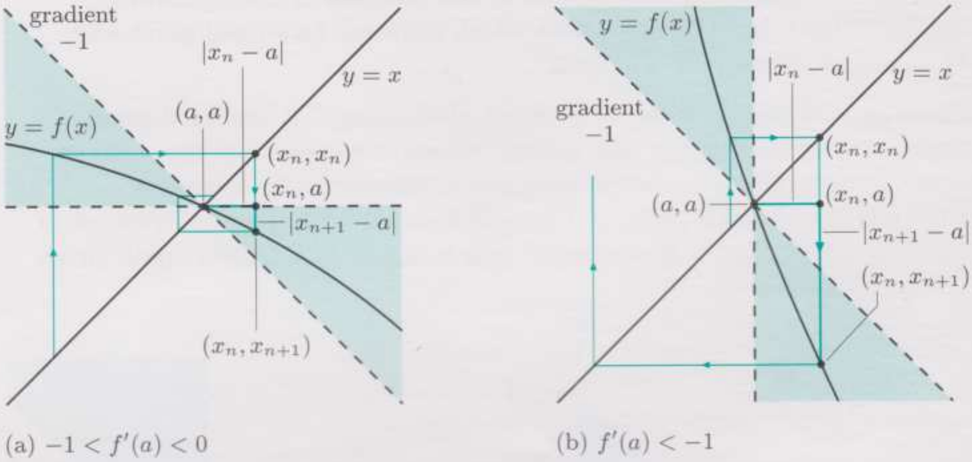


Figure 2.5 Graphs with negative gradient at a

In Figure 2.5(a), the gradient $f'(a)$ satisfies $-1 < f'(a) < 0$, so near the fixed point the graph of $y = f(x)$ lies in the shaded region shown. As a result, the distance $|x_{n+1} - a|$ from x_{n+1} to a is *less than* the distance $|x_n - a|$ from x_n to a, so x_{n+1} is closer to a than is x_n. In Figure 2.5(b), however, where $f'(a) < -1$, the distance $|x_{n+1} - a|$ is greater than the distance $|x_n - a|$, so x_{n+1} is further from a than is x_n.

Combining all these observations, we see that

Note that the single inequality $|f'(a)| < 1$ is equivalent to the two inequalities $-1 < f'(a)$ and $f'(a) < 1$.

- ◇ if $-1 < f'(a) < 1$, that is, $|f'(a)| < 1$, then nearby iteration sequences move towards a;
- ◇ if $f'(a) > 1$ or $f'(a) < -1$, that is, $|f'(a)| > 1$, then nearby iteration sequences move away from a.

This description of the behaviour of iteration sequences near a gives rise to the following definitions. A fixed point a of a smooth function f is called **attracting** if $|f'(a)| < 1$, and **repelling** if $|f'(a)| > 1$.

The above reasoning leads to the following result about the behaviour of iteration sequences near such fixed points.

Behaviour near a fixed point

Let a be a fixed point of a smooth function f, and let x_n be an iteration sequence generated by f.

(a) If $|f'(a)| < 1$, then there is an open interval I containing a with the property that if x_0 is in I, then $x_n \to a$ as $n \to \infty$.

(b) If $|f'(a)| > 1$, then no iteration sequence generated by f converges to a, unless $x_n = a$ for some value of n.

An open interval is an interval that does not include its endpoints.

Part (a) states that if an iteration sequence starts 'close enough' to an attracting fixed point, then the sequence converges to a.
Part (b) states that an iteration sequence converges to a repelling fixed point only if the sequence actually 'lands on' the fixed point.

A fixed point a for which $f'(a) = \pm 1$ is called **indifferent**; the behaviour of iteration sequences near such a fixed point can be difficult to determine. Another case that is usually distinguished is the one where $f'(a) = 0$; this is called a **super-attracting** fixed point because nearby iteration sequences converge to such a fixed point particularly quickly.

To illustrate this classification, consider the function $f(x) = x^2$. The fixed points of f are the solutions of the fixed point equation

$$f(x) = x^2 = x; \quad \text{that is,} \quad x^2 - x = 0.$$

Since $x^2 - x = x(x-1)$, the solutions are $x = 0$ and $x = 1$, so the fixed points of f are 0 and 1, as can be seen in Figure 1.5 (page 11).

Now $f'(x) = 2x$, and hence:

$f'(0) = 2 \times 0 = 0$, so 0 is a super-attracting fixed point;

$f'(1) = 2 \times 1 = 2 > 1$, so 1 is a repelling fixed point.

This classification of the fixed points 0 and 1 agrees with the observations in Activities 1.2 and 1.3 that some iteration sequences generated by $f(x) = x^2$ tend rapidly to 0, and some move *away* from 1.

Activity 2.2 Classifying fixed points

For each of the following functions f, find and classify the fixed points of f, and state whether your classification agrees with the behaviour of iteration sequences observed in Activities 1.4, 1.5 and 1.6.

(a) $f(x) = x^2 + \frac{1}{2}$

(b) $f(x) = x^2 + \frac{1}{8}$

(c) $f(x) = x^2 - \frac{1}{2}$

(d) $f(x) = x^2 - 1$

(e) $f(x) = -\frac{1}{8}x^2 + \frac{11}{8}x + \frac{1}{2}$

Solutions are given on page 51.

Finding intervals of attraction

It was stated above that if a is an attracting fixed point of a function f, then there is an open interval I containing a with the property that if x_0 is in I and x_n is generated by iteration of f, then $x_n \to a$ as $n \to \infty$. Any open interval containing a with this property is called an **interval of attraction** for the attracting fixed point a. If $f'(a) > 0$, then such an interval of attraction can be found graphically. Consider, for example, the function

$$f(x) = -\tfrac{1}{8}x^2 + \tfrac{11}{8}x + \tfrac{1}{2}$$

studied in Activity 1.6, whose graph is shown in Figure 2.6, overleaf.

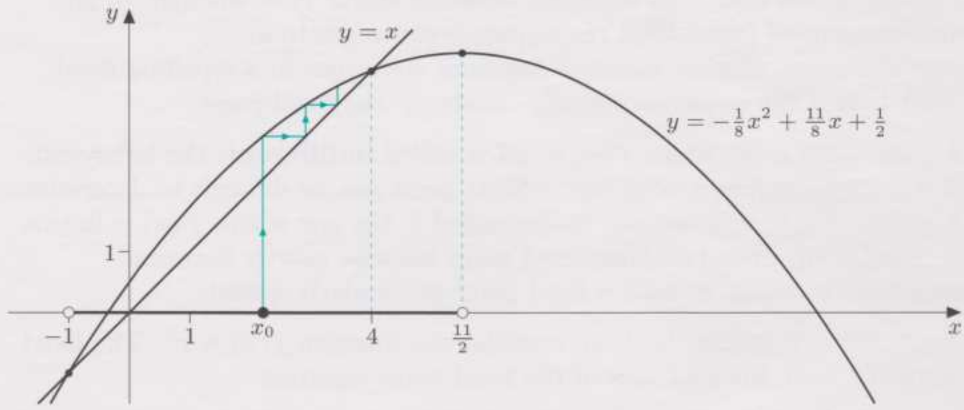


Figure 2.6 An interval of attraction for $f(x) = -\frac{1}{8}x^2 + \frac{11}{8}x + \frac{1}{2}$ and $a = 4$

This function f has an attracting fixed point, $a = 4$, and a repelling fixed point, $a = -1$; see Activity 2.2(e). The function f is increasing on the open interval $(-1, \frac{11}{2})$, and graphical iteration indicates that any iteration sequence x_n generated by f with initial term x_0 in the interval $(-1, \frac{11}{2})$ converges to 4 (this convergence was illustrated for $x_0 = 0$ and for $x_0 = 5$ in Activity 1.6(b)). Therefore $(-1, \frac{11}{2})$ is an interval of attraction for $a = 4$.

Similar reasoning yields the following result, illustrated in Figure 2.7.

Interval of attraction: graphical criterion

Suppose that f is a smooth function with an attracting fixed point a. If I is an open interval on which f is increasing, and a is the only fixed point of f in I, then I is an interval of attraction for a.

If such an interval I is known, then any open subinterval of I which includes the fixed point a is also an interval of attraction for a.

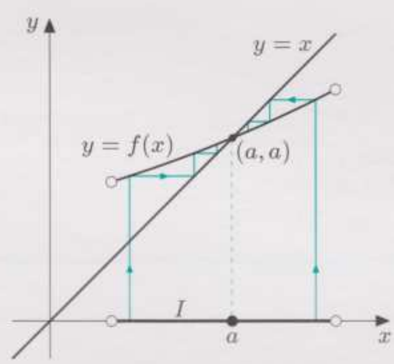


Figure 2.7 Finding an interval of attraction graphically

The next activity gives you a chance to use this graphical criterion.

Activity 2.3 Using the graphical criterion

The function $f(x) = x^2 + \frac{1}{8}$ has an attracting fixed point, $a = \frac{1}{2} - \frac{1}{4}\sqrt{2}$, and a repelling fixed point, $a = \frac{1}{2} + \frac{1}{4}\sqrt{2}$; see Activity 2.2(b). Use the graphical criterion to find an interval of attraction for $\frac{1}{2} - \frac{1}{4}\sqrt{2}$.

A solution is given on page 52.

It is harder to identify an interval of attraction when the function f is decreasing near the attracting fixed point. Here is a general result for smooth functions, which can be proved using the ideas in the discussion after Figure 2.5.

Interval of attraction: gradient criterion

Suppose that f is a smooth function with an attracting fixed point a. If I is an open interval with midpoint a such that

$$|f'(x)| < 1, \quad \text{for } x \in I, \tag{2.1}$$

then I is an interval of attraction for a.

Recall that the symbol $\in$ is read as 'in' or 'belongs to'.

To apply this result for a given smooth function f and attracting fixed point a, we need to *solve* the inequality $|f'(x)| < 1$, that is, determine the set of points x which satisfy this inequality, and then choose an open interval from this set with midpoint a. Here is an example.

Example 2.1 Using the gradient criterion

(a) Sketch the graph of the function $f(x) = -\frac{1}{8}x^2 + \frac{5}{8}x + \frac{7}{2}$.

(b) Find and classify the fixed points of the function f.

(c) Use the gradient criterion to find an interval of attraction I for an attracting fixed point of the function f.

Solution

(a) On completing the square, we obtain

$$f(x) = -\tfrac{1}{8}\left(x - \tfrac{5}{2}\right)^2 + \tfrac{137}{32}.$$

Hence the vertex has coordinates $(\frac{5}{2}, \frac{137}{32})$, and is the highest point of the graph of f.

$\frac{137}{32} \simeq 4.28$

Also, the y-intercept is $f(0) = \frac{7}{2}$, and the x-intercepts are found by solving the equation $f(x) = 0$ to give $x = \frac{1}{2}(5 \pm \sqrt{137})$. Thus the graph of f is as shown in Figure 2.8.

$\frac{1}{2}(5 - \sqrt{137}) \simeq -3.35$
$\frac{1}{2}(5 + \sqrt{137}) \simeq 8.35$

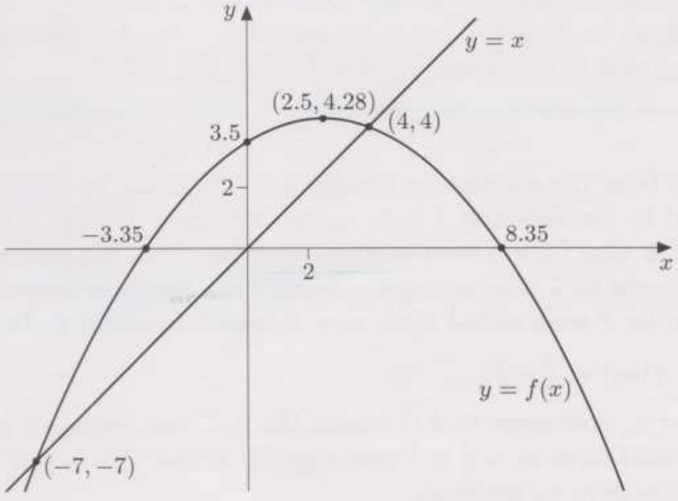


Figure 2.8 The graph of $y = -\frac{1}{8}x^2 + \frac{5}{8}x + \frac{7}{2}$

(b) The fixed point equation is

$$-\tfrac{1}{8}x^2 + \tfrac{5}{8}x + \tfrac{7}{2} = x; \quad \text{that is,} \quad x^2 + 3x - 28 = 0,$$

which has solutions $x = -7$ and $x = 4$. Thus the fixed points of f are -7 and 4, which are also shown in Figure 2.8.

Now, by the quadratic gradient formula (Frame 4),

$$f'(x) = -\tfrac{1}{4}x + \tfrac{5}{8},$$

and hence the gradients at the fixed points -7 and 4 are

$$f'(-7) = -\tfrac{1}{4} \times (-7) + \tfrac{5}{8} = \tfrac{19}{8} \quad \text{and} \quad f'(4) = -\tfrac{1}{4} \times 4 + \tfrac{5}{8} = -\tfrac{3}{8}.$$

So

$$|f'(-7)| = \tfrac{19}{8} > 1 \quad \text{and} \quad |f'(4)| = \tfrac{3}{8} < 1.$$

Thus -7 is a repelling fixed point, and 4 is an attracting fixed point.

(c) The condition $|f'(x)| < 1$ can be written as the two inequalities

$$-1 < f'(x) < 1; \quad \text{that is,} \quad -1 < -\tfrac{1}{4}x + \tfrac{5}{8} < 1.$$

We can solve these two inequalities in turn as follows.

◇ $-\tfrac{1}{4}x + \tfrac{5}{8} < 1$ is equivalent to

$$\tfrac{5}{8} - 1 < \tfrac{1}{4}x; \quad \text{that is,} \quad -\tfrac{3}{2} < x.$$

◇ $-1 < -\tfrac{1}{4}x + \tfrac{5}{8}$ is equivalent to

$$\tfrac{1}{4}x < 1 + \tfrac{5}{8}; \quad \text{that is,} \quad x < \tfrac{13}{2}.$$

Thus the inequality $|f'(x)| < 1$ is equivalent to the two inequalities

$$-\tfrac{3}{2} < x < \tfrac{13}{2}.$$

Hence the set of values of x for which $|f'(x)| < 1$ is the open interval $(-\tfrac{3}{2}, \tfrac{13}{2})$, shown in Figure 2.9.

Figure 2.9 The interval where $|f'(x)| < 1$

Since the attracting fixed point 4 is nearer to the endpoint $\tfrac{13}{2}$ than to $-\tfrac{3}{2}$, we choose I to have right-hand endpoint $\tfrac{13}{2}$. In order that the attracting fixed point 4 is the midpoint of I, to satisfy the gradient criterion, we take the left-hand endpoint to be $4 - (\tfrac{13}{2} - 4) = \tfrac{3}{2}$. Thus an interval of attraction for 4 is $I = (\tfrac{3}{2}, \tfrac{13}{2})$.

Alternatively, you can find the left-hand endpoint, s say, by solving the equation

$$4 = \tfrac{1}{2}(s + \tfrac{13}{2}).$$

It follows from the solution to Example 2.1 that *any* iteration sequence generated by the function f with initial term $x_0 \in I = (\tfrac{3}{2}, \tfrac{13}{2})$ converges to 4. But it *also* follows that certain other iteration sequences generated by f converge to 4. For example, consider the iteration sequence x_n generated by f with initial term $x_0 = 0$, which is *not* in I. In this case,

$$x_1 = f(x_0) = \tfrac{7}{2} \in I,$$

and hence x_n converges to 4 (because the iteration sequence generated by f with initial term $x_1 = \tfrac{7}{2} \in I$ converges to 4, and this is just the sequence x_n with the term x_0 omitted).

Similar reasoning applies whenever the initial term x_0 of an iteration sequence generated by f lies in the half-open interval $(-7, \frac{3}{2}]$, as illustrated in Figure 2.10. Therefore $(-7, \frac{13}{2})$ is a larger interval of attraction for the fixed point 4.

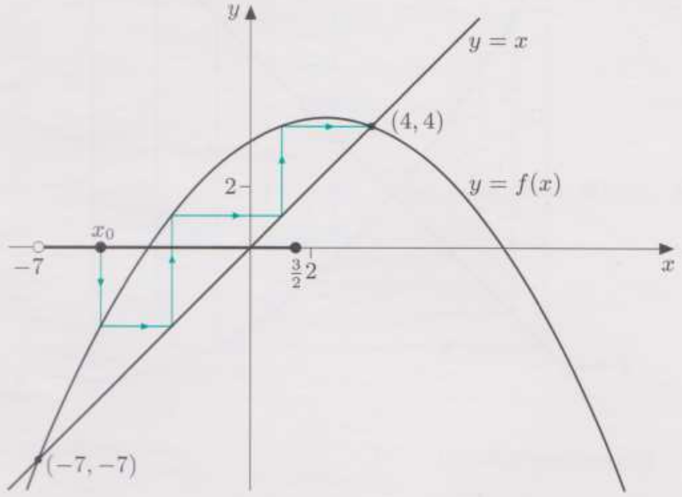


Figure 2.10 A larger interval of attraction

Activity 2.4 Finding an interval of attraction

The function

$$f(x) = x^2 - \tfrac{1}{2}$$

has an attracting fixed point $a = \frac{1}{2} - \frac{1}{2}\sqrt{3}$; see Activity 2.2(c). Use the gradient criterion to find an interval of attraction I for this attracting fixed point.

A solution is given on page 52.

Comment

In the solution to Activity 2.2(c), it was remarked that, on the basis of graphical iteration, the iteration sequence x_n given by

$$x_0 = 0, \quad x_{n+1} = f(x_n) \quad (n = 0, 1, 2, \ldots)$$

appears to converge to the attracting fixed point $\frac{1}{2} - \frac{1}{2}\sqrt{3}$ of f. The initial value 0 does not belong to the interval of attraction $I = (-\frac{1}{2}, \frac{3}{2} - \sqrt{3})$ found in this activity, since $\frac{3}{2} - \sqrt{3} \simeq -0.232$. Also, x_1 does not belong to I, but you can check that $x_2 = -0.25 \in I$, so the sequence x_n does indeed converge to $\frac{1}{2} - \frac{1}{2}\sqrt{3}$.

In Activity 2.2(d), you found that $f(x) = x^2 - 1$ has two fixed points, namely $\frac{1}{2}(1 + \sqrt{5}) \simeq 1.618$ and $\frac{1}{2}(1 - \sqrt{5}) \simeq -0.618$, which are both repelling. So no iteration sequences generated by this function f converge to either of these two fixed points (except those that 'land on' one of them). Nevertheless, some iteration sequences generated by $f(x) = x^2 - 1$ have behaviour which is quite similar to convergence. To see this behaviour, we perform graphical iteration for this function, starting at $x_0 = -\frac{1}{2}$.

Recall that $\frac{1}{2}(1 + \sqrt{5})$ is the golden ratio ϕ (see Chapter A1, Section 1); but this fact has no special significance here.

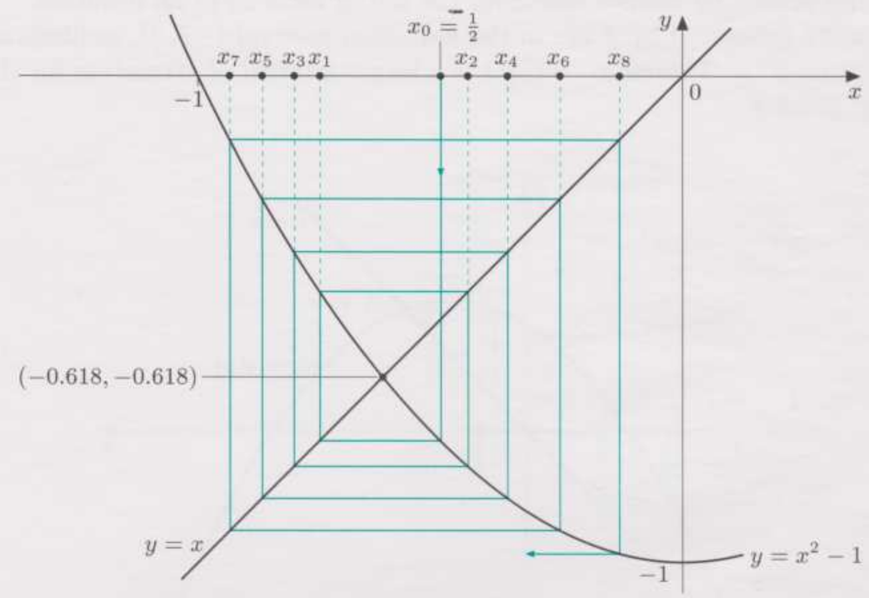


Figure 2.11 Expanding cobweb

Figure 2.11 shows that, as expected, the terms of the sequence x_n move away from the repelling fixed point -0.618. Graphical iteration gives an expanding cobweb in which the terms with even subscripts

$$x_0,\ x_2,\ x_4,\ \ldots$$

form a sequence which appears to tend to 0, whereas the terms with odd subscripts

$$x_1,\ x_3,\ x_5,\ \ldots$$

form a sequence which appears to tend to -1. The numbers 0 and -1 are *not* fixed points of f, but they do have the property that

$$f(0) = -1 \quad \text{and} \quad f(-1) = 0.$$

These two equations imply that

$$f(f(0)) = f(-1) = 0 \quad \text{and} \quad f(f(-1)) = f(0) = -1.$$

So 0 and -1 are both fixed points of a new function with rule $x \longmapsto f(f(x))$, obtained by applying the function f *twice*. We discuss such 'composite' functions in Section 3.

Summary of Section 2

This section has introduced:

- ◇ the idea of the gradient of the graph of a function at a point on the graph;
- ◇ a formula for the gradient of the graph of a quadratic function;
- ◇ a classification of fixed points using the gradient, and a description of the behaviour of nearby iteration sequences;
- ◇ the idea of an interval of attraction and methods of determining an interval of attraction.

Exercises for Section 2

Exercise 2.1

Let f be the function

$$f(x) = \tfrac{1}{8}x^2 - x + 7.$$

(a) Sketch the graph of f.

(b) Find the fixed points of f, mark them on your sketch of the graph of f, and classify them.

(c) Use the graphical criterion to find an interval of attraction for one of the fixed points of f.

(d) Describe the long-term behaviour of iteration sequences given by

$$x_{n+1} = f(x_n) \quad (n = 0, 1, 2, \ldots),$$

for each of the following initial terms.

(i) $x_0 = 0$ (ii) $x_0 = -3.5$

Exercise 2.2

Let f be the function

$$f(x) = 2.5x(1 - x).$$

(a) Sketch the graph of f. (This can be done by applying a y-scaling to the graph of $y = x(1 - x)$ in Exercise 1.1(a).)

(b) Find the fixed points of f, mark them on your sketch of the graph of f, and classify them.

(c) Use the gradient criterion to find an interval of attraction for one of the fixed points of f.

(d) Describe the long-term behaviour of iteration sequences given by

$$x_{n+1} = f(x_n) \quad (n = 0, 1, 2, \ldots),$$

for each of the following initial terms.

(i) $x_0 = 0$ (ii) $x_0 = 0.5$

3 Composition of functions

In Chapter A3, you saw that the result of applying one isometry of the plane followed by another is called the *composition* of these two isometries. In this section, this notion of composition is extended to more general functions, and is used to help explain the occurrence of certain types of behaviour in iteration sequences.

3.1 What is composition of functions?

Consider the two functions

$$f(x) = \sqrt{x} \quad \text{and} \quad g(x) = \sin x. \tag{3.1}$$

We can illustrate the effect of applying these functions one after the other by representing the functions as processors that are linked together as in Figure 3.1.

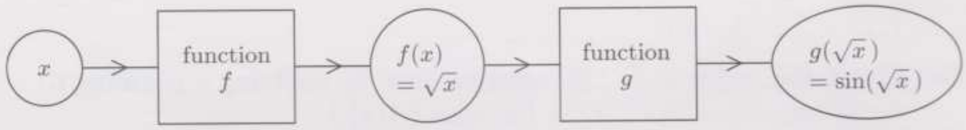


Figure 3.1 A composite processor

When x is fed in on the left, it is processed by f to give $f(x) = \sqrt{x}$, and $\sqrt{x}$ is then processed by g to give $g(\sqrt{x}\,) = \sin(\sqrt{x}\,)$. Thus, overall, the composite processor corresponds to a function with rule

$$x \longmapsto g(f(x)) = \sin(\sqrt{x}\,).$$

The symbol $\circ$ is read as 'oh' or as 'circle'.

This *composite function* is denoted by $g \circ f$.

See Subsection 1.1.

But what are the domain and codomain of this composite function? The domains and codomains of f and g were not specified in equations (3.1), but by our convention the domain of f is $[0, \infty)$, the domain of g is $\mathbb{R}$, and both functions have codomain $\mathbb{R}$. Expressed in two-line notation, we can specify f and g as follows:

$$\begin{aligned} f\colon [0,\infty) &\longrightarrow \mathbb{R} \\ x &\longmapsto \sqrt{x} \end{aligned} \qquad \text{and} \qquad \begin{aligned} g\colon \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto \sin x. \end{aligned}$$

Now, for each x in $[0, \infty)$, the value $f(x) = \sqrt{x}$ is a real number, so it lies in the domain of g. This means that we can form $g(f(x)) = \sin(\sqrt{x}\,)$ for each x in $[0, \infty)$. Therefore we can take $[0, \infty)$ as the domain of $g \circ f$, with $\mathbb{R}$ as the codomain, and write this composite function as

$$\begin{aligned} g \circ f : [0,\infty) &\longrightarrow \mathbb{R} \\ x &\longmapsto \sin(\sqrt{x}\,). \end{aligned}$$

A key point in forming the composite function $g \circ f$ is that

each output value of f is an acceptable input value for g.

This statement can be expressed in terms of the image set of f (that is, the complete set of image values of f) as follows:

See Chapter A3, Section 1.

the image set of f is a subset of the domain of g.

This statement includes the special case when the image set of f is *equal* to the domain of g.

Whenever this statement is true for two functions f and g, we can define the composite of f followed by g.

Definition

Let two functions $f : A \longrightarrow B$ and $g : C \longrightarrow D$ have the property that the image set $f(A)$ is a subset of C. The **composite function** $g \circ f$ is defined by

$$\begin{aligned} g \circ f : A &\longrightarrow D \\ x &\longmapsto g(f(x)). \end{aligned}$$

Here A, B, C and D are any four non-empty sets (not necessarily subsets of $\mathbb{R}$). The notation $f(A)$ for the image set was introduced in Chapter A3, Section 1.

This definition is very general, and allows us to form the composite of a wide variety of functions f and g. The condition that $f(A)$ is a subset of C can be represented diagrammatically as in Figure 3.2. In the caption, the symbol $\subseteq$ has been introduced to denote the phrase 'is a subset of'.

Some texts use $\subset$ rather than $\subseteq$ to denote 'is a subset of'.

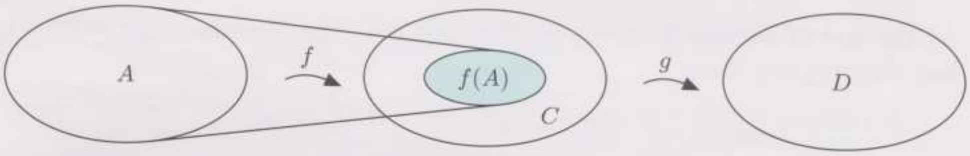


Figure 3.2 The condition that $f(A) \subseteq C$

In this section, we consider only composites of *real* functions. If two real functions f and g both have domain and codomain $\mathbb{R}$, then these functions can be composed. However, according to the above definition, not all pairs of real functions can be composed. For example, with f and g given by equations (3.1), we cannot form the composite function $f \circ g$. The domain of the function $g(x) = \sin x$ is $A = \mathbb{R}$, by our convention, and the image set of g is $g(A) = [-1, 1]$. Since $[-1, 1]$ is *not* a subset of $[0, \infty)$, the domain of the function $f(x) = \sqrt{x}$, we deduce that $f \circ g$ cannot be formed.

Some texts give a more general definition of composition which allows all pairs of functions to be composed.

Example 3.1 Composing functions

Let f and g be the functions

$$f(x) = \tan x \ (x \in [0, \tfrac{1}{2}\pi)) \quad \text{and} \quad g(x) = \sqrt{x}.$$

Show that the composite function $g \circ f$ can be formed, and describe this composite function using two-line notation.

Solution

The domain of f is $A = [0, \frac{1}{2}\pi)$, and its codomain is $B = \mathbb{R}$. The graph of $y = f(x)$ is in Figure 3.3. The image set of f is $f(A) = [0, \infty)$. The domain of g is $C = [0, \infty)$, and its codomain is $D = \mathbb{R}$.

Since $f(A) = C$, the composite function $g \circ f$ exists and is given by

$$\begin{aligned} g \circ f : [0, \tfrac{1}{2}\pi) &\longrightarrow \mathbb{R} \\ x &\longmapsto g(f(x)) = \sqrt{\tan x}. \end{aligned}$$

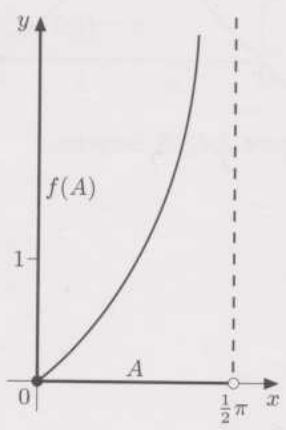


Figure 3.3 The graph of f

Now it is your turn to try composing functions.

Activity 3.1 Composing functions

You will find it convenient to denote the domains and codomains of the three functions f, g and h by A and B, C and D, and E and F, respectively.

Let f, g and h be the functions

$$f(x) = \cos x \quad (x \in (-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi)),$$
$$g(x) = 1/x \quad (x \in (0, \infty)),$$
$$h(x) = \sqrt{x}.$$

(a) Show that the composite function $g \circ f$ can be formed, and describe this composite function using two-line notation.

(b) Show that the composite function $h \circ g$ can be formed, and describe this composite function using two-line notation.

Solutions are given on page 53.

3.2 Cycles and their classification

At the end of Section 2, it was pointed out that the function $f(x) = x^2 - 1$ has the property that

$$f(f(0)) = f(-1) = 0 \quad \text{and} \quad f(f(-1)) = f(0) = -1, \tag{3.2}$$

because $f(0) = -1$ and $f(-1) = 0$. Equations (3.2) can be interpreted as saying that the composite function $f \circ f$ with rule $x \longmapsto f(f(x))$ has the fixed points 0 and -1. For a given function f, any pair of *distinct* points a and b for which

Since f has domain and codomain $\mathbb{R}$, we can form $f \circ f$.

Note that here the symbol a does *not* denote a fixed point of f.

$$f(a) = b \quad \text{and} \quad f(b) = a$$

is said to form a *2-cycle* of f. If we take a as the initial term, then the iteration sequence generated by f is simply

$$a,\ b,\ a,\ b,\ a,\ b,\ \ldots,$$

as illustrated in Figure 3.4. (If b is the initial term, then the sequence is $b, a, b, a, \ldots$.)

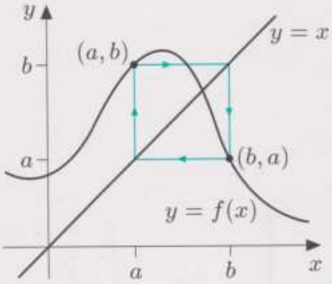


Figure 3.4 A 2-cycle

Definition

Distinct numbers a and b in the domain of a real function f form a **2-cycle** of f if

$$f(a) = b \quad \text{and} \quad f(b) = a. \tag{3.3}$$

So 0 and -1 form a 2-cycle of the function $f(x) = x^2 - 1$.

If equations (3.3) hold, then

$$f(f(a)) = f(b) = a \quad \text{and} \quad f(f(b)) = f(a) = b,$$

so both a and b are fixed points of the composite function $f \circ f$; that is, they are solutions of the equation

$$f(f(x)) = x,$$

which is called the **2-cycle equation** for f.

However, the solutions of the 2-cycle equation need not all belong to a 2-cycle. For if a is a fixed point of f, then

$$f(a) = a, \quad \text{so} \quad f(f(a)) = f(a) = a,$$

and hence a is a solution of the 2-cycle equation, but a does not belong to a 2-cycle.

In general, the solutions of the 2-cycle equation $f(f(x)) = x$ are either fixed points of f or members of 2-cycles of f.

Activity 3.2 Checking 2-cycles

For each of the following functions f, show that the given values of a and b form a 2-cycle of f.

(a) $f(x) = x^2 - \frac{13}{16}$, $\quad a = -\frac{1}{4}, b = -\frac{3}{4}$.

(b) $f(x) = x^2 - 2$, $\quad a = -\frac{1}{2} + \frac{1}{2}\sqrt{5}, b = -\frac{1}{2} - \frac{1}{2}\sqrt{5}$.

Note that you should keep a and b in surd form in this calculation.

Solutions are given on page 53.

At the end of Section 2, you also saw that the iteration sequence

$$x_0 = -\tfrac{1}{2}, \quad x_{n+1} = x_n^2 - 1 \quad (n = 0, 1, 2, \ldots) \tag{3.4}$$

appears to approach the 2-cycle formed by 0 and -1, in the sense that

See Figure 2.11 (page 30).

the sequence $x_0, x_2, x_4, \ldots$ tends to 0,
the sequence $x_1, x_3, x_5, \ldots$ tends to -1.

This is an example of the following general result.

2-Cycle Rule

Suppose that the sequence x_n is generated by iteration of the real function f, and that

the sequence $x_0, x_2, x_4, \ldots$ tends to a,
the sequence $x_1, x_3, x_5, \ldots$ tends to b,

where $a \neq b$. If f is continuous, then a and b form a 2-cycle of the function f.

In this situation, we say that the iteration sequence x_n **tends to** the 2-cycle a, b.

It is convenient to speak of 'the 2-cycle a, b', rather than 'the 2-cycle formed by a and b'.

The notion of an attracting or repelling 2-cycle can also be defined. Consider once again the function $f(x) = x^2 - 1$, which has 2-cycle $0, -1$. Both 0 and -1 are fixed points of the composite function $f \circ f$, which has rule

$$\begin{aligned}(f \circ f)(x) = f(f(x)) &= (x^2 - 1)^2 - 1 \\ &= (x^4 - 2x^2 + 1) - 1 \\ &= x^4 - 2x^2.\end{aligned}$$

The fixed points 0 and -1 of $f \circ f$ can be classified as attracting, repelling, indifferent or super-attracting according to the values of the gradient of the graph of $f \circ f$ at 0 and -1. These gradients are written as

$$(f \circ f)'(0) \quad \text{and} \quad (f \circ f)'(-1).$$

So far, you have not seen how to calculate such values, since $f \circ f$ is not a quadratic function. The graph of $f \circ f$, sketched in Figure 3.5, is helpful in *seeing* what the values of $(f \circ f)'(0)$ and $(f \circ f)'(-1)$ are.

You would not be expected to sketch this graph by hand, at this stage of the course.

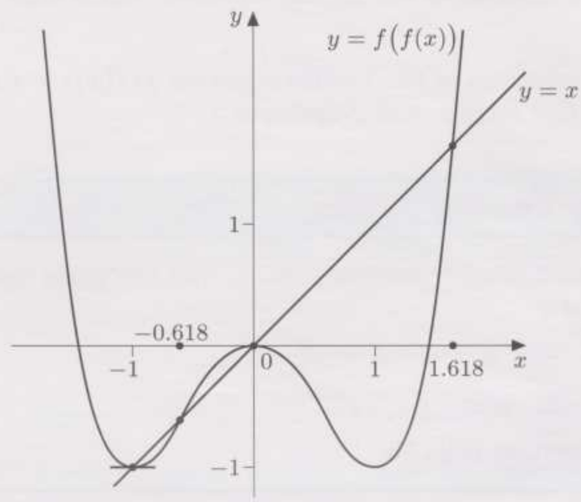


Figure 3.5 Graph of $y = x^4 - 2x^2$

This graph of $y = f(f(x))$ crosses the line $y = x$ four times, so $f \circ f$ has four fixed points. Two of these crossings occur when $x = 0$ and $x = -1$, the 2-cycle of f, and the other two occur when $x = \frac{1}{2}(1 + \sqrt{5}) \simeq 1.618$ and $x = \frac{1}{2}(1 - \sqrt{5}) \simeq -0.618$, the fixed points of f.

See Activity 2.2(d).

In Figure 3.5, it can be seen that the graph has horizontal tangents at $(0, 0)$ and $(-1, -1)$. Hence the gradients of the graph at 0 and -1 both have the value 0, so 0 and -1 are both super-attracting fixed points of $f \circ f$.

It turns out that whenever a, b is a 2-cycle of a smooth function f, the gradients of the graph of the composite function $f \circ f$ at a and b are *the same*, the common value being the product $f'(a)f'(b)$; that is,

$$(f \circ f)'(a) = (f \circ f)'(b) = f'(a)f'(b).$$

So a and b are the same type of fixed point of $f \circ f$. Therefore we say that a 2-cycle a, b of a smooth function f is **attracting**, **repelling**, **indifferent** or **super-attracting** if a and b are attracting, repelling, indifferent or super-attracting, respectively, when considered as fixed points of the composite function $f \circ f$. Thus a 2-cycle a, b of a function f is

attracting	if $\lvert f'(a)f'(b)\rvert < 1$,
repelling	if $\lvert f'(a)f'(b)\rvert > 1$,
indifferent	if $\lvert f'(a)f'(b)\rvert = 1$,
super-attracting	if $f'(a)f'(b) = 0$.

For example, you saw above that $0, -1$ is a 2-cycle of the function $f(x) = x^2 - 1$. In this case $f'(x) = 2x$, $a = 0$ and $b = -1$, so

$$f'(a)f'(b) = (2a)(2b) = 0.$$

Thus $0, -1$ is a super-attracting 2-cycle of f.

The following descriptions of the behaviour of iteration sequences near an attracting or repelling 2-cycle can be deduced by applying to $f \circ f$ the result in Section 2 on behaviour near an attracting or repelling fixed point.

See page 24.

Behaviour near an attracting or repelling 2-cycle

Let a, b be a 2-cycle of the smooth function f, and let x_n be an iteration sequence generated by f.

(a) If $|f'(a)f'(b)| < 1$, then there is an open interval I containing a with the property that if x_0 is in I, then

the sequence $x_0, x_2, x_4, \ldots$ tends to a,
the sequence $x_1, x_3, x_5, \ldots$ tends to b.

(b) If $|f'(a)f'(b)| > 1$, then x_n does not tend to the 2-cycle a, b unless $x_n = a$ for some value of n.

Part (a) states that if an iteration sequence starts close enough to one member of an attracting 2-cycle, then the sequence tends to the 2-cycle.

If $x_n = a$, then $x_{n+1} = b$, $x_{n+2} = a$, and so on.

The next activity gives you a chance to classify 2-cycles and relate this classification to the behaviour of nearby iteration sequences.

Activity 3.3 Classifying 2-cycles

For each of the following functions f, classify the given 2-cycle a, b. Also, in each case calculate, correct to three significant figures, the first 6 terms of the iteration sequence x_n generated by f with the given initial term x_0, and relate the behaviour of these terms to the classification of the 2-cycle.

(a) $f(x) = x^2 - \frac{13}{16}$, $\quad a = -\frac{1}{4}, b = -\frac{3}{4}$, $\quad x_0 = 0$.

(b) $f(x) = x^2 - 2$, $\quad a = -\frac{1}{2} + \frac{1}{2}\sqrt{5}, b = -\frac{1}{2} - \frac{1}{2}\sqrt{5}$, $\quad x_0 = \frac{1}{2}$.

You checked these 2-cycles in Activity 3.2.

Solutions are given on page 53.

Activity 3.3 (and Exercise 3.2 below) should convince you that it is in no sense 'strange' for an iteration sequence to tend to a 2-cycle. However, it is often difficult to *find* 2-cycles of a function f by hand calculation, since this involves solving the 2-cycle equation

$$f(f(x)) = x,$$

and the rule for $f \circ f$ is often complicated. We return to this problem in Section 4, where the computer is used to study the long-term behaviour of iteration sequences, including approach to fixed points, 2-cycles and cycles of higher order.

Summary of Section 3

This section has introduced:

◇ the construction of composite functions, and the process of expressing functions as composites of simpler functions;

◇ the concept of a 2-cycle of a real function f, and its interpretation in terms of fixed points of the composite function $f \circ f$;

◇ the 2-Cycle Rule, the classification of 2-cycles, and the behaviour of nearby iteration sequences.

Exercises for Section 3

Exercise 3.1

For each of the following pairs of functions f and g, show that the composite function $g \circ f$ can be formed, and describe this composite function using two-line notation.

(a) $f(x) = x^2 + 1, \quad g(x) = \ln x$

(b) $f(x) = \sqrt{x+1}, \quad g(x) = \sqrt{x+1}$

Exercise 3.2

For each of the following functions f, show that the given values a and b form a 2-cycle of f, and classify this 2-cycle.

(a) $f(x) = -x^2 + 2x + 1, \quad a = 1, b = 2.$

(b) $f(x) = 3.2x(1-x), \quad a = \frac{1}{32}(21 + \sqrt{21}\,), b = \frac{1}{32}(21 - \sqrt{21}\,).$

4 Iterating real functions with the computer

In this section, you will need computer access, the files for this chapter and Computer Book B.

The computer can be used in several ways to explore iteration sequences. For example, it can be used to solve the fixed point equation and the 2-cycle equation, to calculate many terms of an iteration sequence, and to perform graphical iteration. This makes it possible to investigate systematically the behaviour of *families* of iteration sequences, such as

$$x_0 = 0, \quad x_{n+1} = x_n^2 + c \quad (n = 0, 1, 2, \ldots),$$

where c is a parameter. As the parameter c varies, various types of long-term behaviour of the sequence x_n are observed, some familiar and some new. For example, in the case $c = -1.76$, the sequence x_n is plotted in Figure 4.1 using graphical iteration of the function $f(x) = x^2 - 1.76$. This sequence x_n seems to tend to *three* numbers (approximately 1.3, 0 and -1.8), which form a 3-cycle of f.

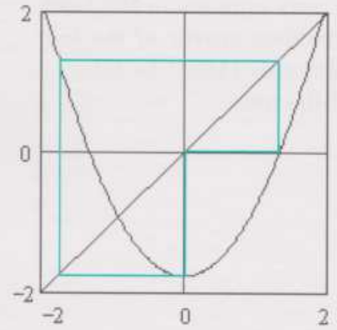


Figure 4.1 The sequence x_n with $c = -1.76$

Definition

Distinct numbers $a_1, a_2, \ldots, a_p$ in the domain of a real function f form a ***p*-cycle** of f if

$$f(a_1) = a_2, \quad f(a_2) = a_3, \quad \ldots, \quad f(a_p) = a_1. \tag{4.1}$$

Let μ be the gradient product $f'(a_1)f'(a_2)\cdots f'(a_p)$. Then the p-cycle is **attracting** if $|\mu| < 1$, **repelling** if $|\mu| > 1$, **indifferent** if $|\mu| = 1$, and **super-attracting** if $\mu = 0$.

These definitions generalise the definitions given earlier for a fixed point and 2-cycle, and their classifications.

Here μ is the Greek letter mu.

For example, the function $f(x) = x^2 - 1.76$ has a 3-cycle: $a_1 \simeq 1.335\,601$, $a_2 \simeq -0.023\,830$ and $a_3 \simeq -1.759\,432$. Also, $f'(x) = 2x$, so

$$\mu = f'(a_1)f'(a_2)f'(a_3) = (2a_1)(2a_2)(2a_3) \simeq -0.45 < 1.$$

In this example, $f(a_1) = a_2$, $f(a_2) = a_3$ and $f(a_3) = a_1$.

Thus this 3-cycle is attracting, as suggested by Figure 4.1.

If equations (4.1) hold, then $a_1, a_2, \ldots, a_p$ all satisfy the equation

$$f(f(\ldots f(x)\ldots)) = x, \quad \text{in which } f \text{ is applied } p \text{ times.}$$

Thus $a_1, a_2, \ldots, a_p$ are all fixed points of the composite function $f \circ f \circ \ldots \circ f$, obtained when the function f is applied p times. This function is called the **pth iterate** of f, often denoted by f^p. The classification of the p-cycle $a_1, a_2, \ldots, a_p$ given above corresponds to the classification of $a_1, a_2, \ldots, a_p$ as fixed points of the function f^p. In particular, it follows that if $a_1, a_2, \ldots, a_p$ form an attracting p-cycle of f, then any iteration sequence generated by f which starts close enough to a member of the cycle must tend to the cycle. The sequence plotted in Figure 4.1 is an example with $p = 3$.

For example, $f^2 = f \circ f$.

Refer to Computer Book B for the work in this section.

Summary of Section 4

This section has used the computer to explore iteration sequences.

5 The Binomial Theorem

In this final section we look at a much older iterative method, one that can be traced back at least to the years 1100–1300AD. In that era, mathematics textbooks in China and Persia sometimes included a large triangle of positive integers, known as the *arithmetical triangle*, which starts as in Figure 5.1.

An arithmetical triangle with 9 rows appears in the book *Precious mirror of the four elements* (1303) by Chu Shih-chieh.

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

Figure 5.1 Pascal's triangle (the arithmetical triangle)

The triangle is constructed with 1s on the left and right edges, and each inner entry is the sum of the two nearest entries in the row above; for example, in the fifth row

$$4 = 1 + 3, \quad 6 = 3 + 3, \quad 4 = 3 + 1.$$

Thus the arithmetical triangle can be considered to result from an iteration, since each row, except the first, is derived from the row above it by applying the same process.

In Europe, the triangle in Figure 5.1 first appeared around 1500 in German textbooks on calculation, and it was then studied by various mathematicians, including Blaise Pascal who wrote a treatise on the arithmetical triangle. But why is **Pascal's triangle**, as it is now known, so important?

Pascal (1623–1662) was a French mathematician, physicist and theologian. He also proved results on conics, built a calculating machine and helped to determine the weight of air.

5.1 Expanding binomial expressions

Pascal's triangle is a very convenient way to organise the coefficients which occur when we calculate the expansions of expressions of the form

$$(a+b)^2, \quad (a+b)^3, \quad (a+b)^4, \quad \ldots .$$

For example:

$$(a+b)^2 = a^2 + 2ab + b^2 \quad \text{has coefficients } 1, 2, 1;$$

$$\begin{aligned}(a+b)^3 &= (a+b)(a^2 + 2ab + b^2)\\ &= a^3 + 3a^2b + 3ab^2 + b^3 \quad \text{has coefficients } 1, 3, 3, 1;\end{aligned}$$

$$\begin{aligned}(a+b)^4 &= (a+b)(a^3 + 3a^2b + 3ab^2 + b^3)\\ &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \quad \text{has coefficients } 1, 4, 6, 4, 1.\end{aligned}$$

Also,

$$(a+b)^1 = a + b$$

has coefficients 1, 1.

The coefficients in these three expansions are the entries in the third, fourth and fifth rows of Pascal's triangle.

The reason why these coefficients appear in Pascal's triangle is not hard to find. For instance, the term $10a^3b^2$ in the expansion of $(a+b)^5$ arises as the sum of $4a^3b^2$ and $6a^3b^2$ from the product of $(a+b)$ with $(a+b)^4$:

$$4a^3b^2$$

$$(a+b)^5 = (a+b)(a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4)$$

$$6a^3b^2$$

$$= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

Similarly, every inner term in the expansion of $(a+b)^n$, where $n > 1$, is the sum of two terms, each corresponding to a term in the expansion of $(a+b)^{n-1}$. Inspection shows that this sum is the one used to construct Pascal's triangle.

An expression of the form $(a+b)^n$ is called a **binomial expression**, and its expansion is called a **binomial expansion**. The binomial expansion of $(a+b)^n$ can be used to find more complicated expansions of the same type. For example, setting $a = 3$ and $b = -2x$ in

The word 'binomial' indicates that two variables are involved.

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4,$$

we obtain

$$\begin{aligned}(3-2x)^4 &= 3^4 + 4 \times 3^3 \times (-2x) + 6 \times 3^2 \times (-2x)^2 \\ &\quad + 4 \times 3 \times (-2x)^3 + (-2x)^4 \\ &= 81 - 216x + 216x^2 - 96x^3 + 16x^4.\end{aligned}$$

Activity 5.1 Finding expansions

(a) Calculate the two rows of Pascal's triangle which follow those shown in Figure 5.1.

(b) Use part (a) to expand each of the following binomial expressions.

(i) $(a+b)^6$ (ii) $(1+x)^7$ (iii) $(1+2x)^6$

(c) Expand the binomial expression $(2 - \frac{1}{2}x)^5$.

Solutions are given on page 54.

Pascal's triangle is a convenient way to obtain the coefficients in the expansions of $(a+b)^n$ for fairly small values of n. However, it would be tedious to obtain a particular coefficient in the expansion of $(a+b)^{12}$, say, by using Pascal's triangle. It is therefore desirable to have a *closed-form* formula.

A formula for the coefficients in the expansion of $(a+b)^n$ was known to early Persian mathematicians; a proof that this formula is correct was given by Pascal. It turns out that these coefficients are related to a certain 'counting problem'. To see why this is, we write $(a+b)^n$ in the form

For example, Al-Kashi, who died in Samarkand in 1429, gave this formula in his book *Key of arithmetic*.

$$(a+b)^n = \underbrace{(a+b) \times (a+b) \times \cdots \times (a+b)}_{n \text{ pairs of brackets}}. \qquad (5.1)$$

The expansion of the right-hand side of equation (5.1) is a sum of terms of the form

$$a^n, a^{n-1}b, a^{n-2}b^2, \dots, b^n; \quad \text{that is,} \quad a^{n-k}b^k \text{ for } k = 0, 1, \dots, n.$$

Each term of the form $a^{n-k}b^k$ arises by choosing the variable b from k of the pairs of brackets on the right-hand side of equation (5.1) and the variable a from the remaining $n-k$ pairs of brackets. So the coefficient of $a^{n-k}b^k$ is equal to the number of different ways of choosing k pairs of brackets from the n pairs of brackets. For example, the coefficient of a^3b in the expansion of $(a+b)^4$ is 4 because there are 4 ways of choosing one pair of brackets from the 4 pairs of brackets.

Thus we need to find a formula for the number of ways of choosing k objects from a set of n objects. The next subsection is devoted to this counting problem.

5.2 Permutations and combinations

This subsection introduces methods of counting various arrangements of and selections from a number of objects. A **permutation** is an arrangement of objects (all different) in a particular order. It is natural to ask how many permutations there are of n different objects. Here is an example to show how permutations are counted.

Example 5.1 Permutations of 1, 2, 3

Each three-digit number corresponds to a different permutation of $1, 2, 3$.

How many three-digit numbers can be made using the digits $1, 2, 3$ exactly once each?

Solution

Here is a systematic way to count how many such numbers can be made. There are 3 choices for the first digit, then 2 choices for the second digit, and finally 1 choice for the third digit. This gives $3 \times 2 \times 1 = 6$ three-digit numbers, which are displayed below:

The numbers have been displayed systematically: in numerical order with each column having entries with the same first digit.

123, 213, 312,
132, 231, 321.

In general, suppose that we are required to count the number of ways that n objects can be arranged. Then

- ◇ the first object can be chosen in n ways;
- ◇ the second object can be chosen in $n-1$ ways;

  ⋮

- ◇ the nth object can be chosen in 1 way.

Thus the number of possible permutations of n different objects is

$$n \times (n-1) \times \cdots \times 2 \times 1. \tag{5.2}$$

This product of the first n positive integers is denoted by the symbol $n!$ and it is called ***n* factorial** or **factorial *n***.

For example, 10 different objects can be arranged in

$$10! = 10 \times 9 \times \cdots \times 2 \times 1 = 3\,628\,800$$

different ways. As this calculation suggests, the values of $n!$ increase rapidly as n increases. The first few values are shown in Table 5.1.

Many calculators can be used to evaluate $n!$ (often for $n = 0, 1, 2, \ldots, 69$).

Table 5.1 Values of $n!$

n	0	1	2	3	4	5	6	7	8	9	10
$n!$	1	1	2	6	24	120	720	5040	40 320	362 880	3 628 800

As indicated in the table, it is conventional to define

$$0! = 1. \tag{5.3}$$

Notice that, with this convention, we have

$$1! = 1 \times 0!, \quad 2! = 2 \times 1!, \quad 3! = 3 \times 2!, \quad \text{and so on.}$$

In general, $(n+1)!$ can be expressed in terms of $n!$ as

$$(n+1)! = (n+1)n!, \quad \text{for } n = 0, 1, 2, \ldots.$$

Activity 5.2 Permuting letters

(a) How many permutations are there of the letters A, B, C, D?

(b) Construct a table showing all the permutations of A, B, C, D in a systematic fashion using four columns.

Solutions are given on page 54.

Permutations of a list of letters are written without spaces; for example, ABCD and BCAD are permutations of A, B, C, D.

Next, we consider a more complicated problem. An arrangement formed by choosing k objects from n objects (all different) and placing them in a particular order is called a **permutation of n objects taken k at a time**. How many such permutations are there? Here is an example.

Example 5.2 Two-letter permutations

(a) How many two-letter permutations can be formed from A, B, C, D, E?

(b) Construct a systematic table with five columns showing all the permutations in part (a).

Solution

(a) The first letter can be chosen in 5 ways, and then the second letter can be chosen in 4 ways. Thus

$$5 \times 4 = 20$$

two-letter permutations can be formed from A, B, C, D, E.

(b) The 20 permutations are displayed below using 'dictionary order' in each column.

AB	BA	CA	DA	EA
AC	BC	CB	DB	EB
AD	BD	CD	DC	EC
AE	BE	CE	DE	ED

The permutation AB is different from BA.

In general, suppose that we are required to arrange k objects chosen from n objects. Then

◇ the first object can be chosen in n ways;

◇ the second object can be chosen in $n-1$ ways;

⋮

◇ the kth object can be chosen in $n-(k-1)=n-k+1$ ways.

Thus the number of permutations of n objects taken k at a time is

$$n \times (n-1) \times \cdots \times (n-k+1) = n(n-1)\cdots(n-k+1),$$

which is the product of the k largest terms in the expansion of $n!$. This product is denoted by the symbol ${}^nP_k$. It can be expressed in closed form by introducing the expression $(n-k)!$ in the numerator and denominator, as follows:

The symbol ${}^nP_k$ is read as 'n P k'.

$$\begin{aligned} {}^nP_k &= n(n-1)\cdots(n-k+1) && (5.4)\\ &= \frac{n(n-1)\cdots(n-k+1)(n-k)!}{(n-k)!}\\ &= \frac{n!}{(n-k)!}. && (5.5)\end{aligned}$$

Equation (5.5) provides a convenient closed-form formula for ${}^nP_k$, but when *calculating* a given ${}^nP_k$, equation (5.4) is often the one to use.

Activity 5.3 Arranging five letters from eight

How many five-letter permutations can be made from eight different letters?

A solution is given on page 54.

Now we return to the counting problem introduced at the end of Subsection 5.1. A selection of k objects from n objects (all different) in which order does *not* matter is called a **combination of n objects taken k at a time**. How many such combinations are there? Once again, here is an example.

The number of such combinations was known to Hindu mathematicians as early as 850AD.

Example 5.3 Two-letter combinations

How many two-letter combinations are there of the letters A, B, C, D, E?

Solution

First note that each two-letter combination gives *two* two-letter permutations; for example, the combination A,B gives the permutations AB and BA. In Example 5.2(a), we found that there are 20 two-letter permutations from A, B, C, D, E, as displayed in the solution to Example 5.2(b). Therefore there are $\frac{1}{2} \times 20 = 10$ two-letter combinations of A, B, C, D, E, as displayed below.

The use here of commas between the letters in a combination indicates that the order does not matter.

A,B			
A,C	B,C		
A,D	B,D	C,D	
A,E	B,E	C,E	D,E

In general, any combination of k different objects (in which order does not matter) can be arranged in $k!$ ways, for there are $k!$ permutations of the objects. So the number of combinations of n objects taken k at a time is obtained by dividing ${}^nP_k$ by $k!$. Thus if we denote the number of combinations of n objects taken k at a time by the symbol ${}^nC_k$, then

$$ {}^nC_k = \frac{{}^nP_k}{k!} = \frac{n(n-1)\cdots(n-k+1)}{k!}, \tag{5.6} $$

so

$$ {}^nC_k = \frac{n!}{(n-k)!\,k!}, \tag{5.7} $$

in view of equation (5.5). For example, the number of two-letter combinations from A, B, C, D, E is

$$ {}^5C_2 = \frac{5 \times 4}{2 \times 1} = 10 \quad \text{(by equation (5.6))}, $$

as we found in Example 5.3.

The symbol ${}^nC_k$ is read as 'n C k' or 'n choose k'.

Alternatively, by equation (5.7),

$$ \begin{aligned} {}^5C_2 &= \frac{5!}{3!\,2!} \\ &= \frac{5 \times 4 \times 3 \times 2 \times 1}{(3 \times 2 \times 1)(2 \times 1)} \\ &= 10. \end{aligned} $$

Activity 5.4 Counting combinations

(a) In how many ways can a subcommittee of 4 be selected from a committee of 10 people? (Note that the members of a subcommittee are not ordered.)

(b) How many 6-number combinations can be selected from the numbers $1, 2, \ldots, 49$?

You may have made such a selection in a lottery!

Solutions are given on page 54.

This subsection concludes with several remarks about ${}^nC_k$.

1. The notation

$$ {}^nC_k = \frac{n!}{(n-k)!\,k!} $$

does make sense even when $k = 0$ and $k = n$, in view of the convention that $0! = 1$. For example,

$$ {}^5C_5 = \frac{5!}{0!\,5!} = 1 $$

is indeed the number of ways of selecting 5 objects from 5 objects.

2. It can be seen from equation (5.6) that ${}^nC_{k+1}$ can be obtained by multiplying ${}^nC_k$ by the fraction $\dfrac{n-k}{k+1}$. For example, with $k = 2$,

$$ \begin{aligned} {}^nC_3 = \frac{n(n-1)(n-2)}{3!} &= \frac{n(n-1)}{2!} \times \frac{n-2}{3} \\ &= {}^nC_2 \times \frac{n-2}{3}. \end{aligned} $$

3. In general, we have

$$ {}^nC_k = {}^nC_{n-k}, $$

by equation (5.7). For example, ${}^5C_2 = 10 = {}^5C_3$.

5.3 Binomial coefficients

In Subsection 5.1 we observed that in the expansion of $(a+b)^n$ the coefficient of $a^{n-k}b^k$ is equal to the number of ways of selecting k pairs of brackets from n pairs of brackets. In Subsection 5.2 we found that this number equals

$$^nC_k = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{(n-k)!\,k!}.$$

Thus we have established the following result, giving the expansion of $(a+b)^n$.

Note that the coefficients of a^n and b^n could be written as

$$^nC_0 = 1 \quad \text{and} \quad ^nC_n = 1,$$

respectively.

Binomial Theorem

For $n = 1, 2, 3, \ldots,$

$$(a+b)^n = a^n + {}^nC_1 a^{n-1}b + \cdots + {}^nC_k a^{n-k}b^k + \cdots + b^n.$$

For example, the coefficient of a^8b^4 in the expansion of $(a+b)^{12}$ is

$$^{12}C_4 = \frac{12 \times 11 \times 10 \times 9}{4 \times 3 \times 2 \times 1} = 495.$$

A common alternative notation for the binomial coefficients is

$$\binom{n}{k},$$

which is read as 'n choose k'.

Because they appear in the binomial expansion, the numbers ^nC_k are called **binomial coefficients**. It is useful to be familiar with the first few binomial coefficients in the expansion of $(a+b)^n$:

$$^nC_0 = 1, \quad ^nC_1 = n, \quad ^nC_2 = \frac{n(n-1)}{2!}, \quad ^nC_3 = \frac{n(n-1)(n-2)}{3!}.$$

An important special case of the Binomial Theorem is obtained by taking $a = 1$ and $b = x$. This gives, for $n = 1, 2, 3, \ldots,$

$$\begin{aligned}(1+x)^n &= 1 + {}^nC_1x + {}^nC_2x^2 + \cdots + {}^nC_kx^k + \cdots + x^n \\ &= 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + x^n.\end{aligned}$$

When calculating successive binomial coefficients, it is helpful to use remarks 2 and 3 at the end of Subsection 5.2.

Activity 5.5 Using the Binomial Theorem

(a) Use the Binomial Theorem to find the first four terms in the expansion of each of the following expressions.

(i) $(1+x)^{10}$ (ii) $(2+\frac{1}{3}x)^{10}$

(b) Find the coefficient of a^6b^5 in the expansion of $(a+b)^{11}$.

(c) Find the constant term in the expansion of $\left(x - \frac{1}{x}\right)^{12}$.

Solutions are given on page 54.

Comment

Here we use the factorisation

$$2 + \tfrac{1}{3}x = 2(1 + \tfrac{1}{6}x).$$

In part (a)(ii), you may prefer to use the following approach:

$$\begin{aligned}(2+\tfrac{1}{3}x)^{10} &= 2^{10}(1+\tfrac{1}{6}x)^{10} \\ &= 2^{10}(1 + {}^{10}C_1(\tfrac{1}{6}x) + {}^{10}C_2(\tfrac{1}{6}x)^2 + \cdots).\end{aligned}$$

Summary of Section 5

This section has introduced:

- ◇ Pascal's triangle and its connection with the expansion of $(a+b)^n$;
- ◇ permutations and combinations;
- ◇ the expression

$$^{n}P_k = n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

for the number of permutations of n objects taken k at a time;

- ◇ the expression

$$^{n}C_k = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{(n-k)!\,k!}$$

for the number of combinations of n objects taken k at a time;

- ◇ the Binomial Theorem: for $n = 1, 2, 3, \ldots$,

$$(a+b)^n = a^n + {}^{n}C_1 a^{n-1}b + \cdots + {}^{n}C_k a^{n-k}b^k + \cdots + b^n.$$

Exercises for Section 5

Exercise 5.1

Use Pascal's triangle to expand the expression $(1+x)^8$.

Exercise 5.2

(a) How many six-letter permutations can be formed from the letters A, B, C, D, E, F, G, H, I, J?

(b) How many six-letter combinations can be formed from the letters A, B, C, D, E, F, G, H, I, J?

Exercise 5.3

Find the coefficient of x^6 in the expansions of each of the following expressions.

(a) $(1+2x)^6$ (b) $(2-3x)^{10}$ (c) $(5-x^2)^5$

Exercise 5.4

Find the first three terms in the expansions of each of the following expressions.

(a) $(a+b)^{12}$ (b) $(3+\frac{1}{2}x)^{12}$

Summary of Chapter B1

In this chapter, you met several techniques for understanding the long-term behaviour of iteration sequences. You also saw how to calculate the expansions of expressions of the form $(a+b)^n$, either by Pascal's triangle or by use of the Binomial Theorem.

Learning outcomes

You have been working towards the following learning outcomes.

Terms to know and use

Iteration sequence, sequence generated by a function, graphical iteration, staircase, cobweb, fixed point, fixed point equation, smooth function, gradient (or slope), increasing/decreasing function, attracting/repelling/super-attracting/indifferent fixed point, interval of attraction, composite function, 2-cycle, p-cycle, attracting/repelling/super-attracting/indifferent p-cycle, pth iterate, Pascal's triangle, binomial coefficient, binomial expansion, permutation, combination, factorial, Binomial Theorem.

Notation to know and use

$\in$, $f'(x)$, $g \circ f$, $\subseteq$, $(g \circ f)'(a)$, f^p, $n!$, ${}^nP_k$, ${}^nC_k$.

Mathematical skills

- ◇ Calculate a few terms of an iteration sequence by hand.
- ◇ Sketch iteration sequences using graphical iteration.
- ◇ Use the fixed point equation to find fixed points of quadratic functions.
- ◇ Classify fixed points as attracting, repelling, indifferent or super-attracting, and use this information to help determine the behaviour of nearby iteration sequences.
- ◇ Find an interval of attraction for an attracting fixed point.
- ◇ Form the composite of two real functions, where possible.
- ◇ Classify 2-cycles as attracting, repelling, indifferent or super-attracting, and use this information to help determine the long-term behaviour of nearby iteration sequences.
- ◇ Calculate the numbers of possible permutations and combinations.
- ◇ Find the terms in the expansions of expressions of the form $(a+b)^n$.

Mathcad skills

- ◇ Plot graphs of iterates of a function, use these to perform graphical iteration, and interpret the results.
- ◇ Use Mathcad to find and classify fixed points and cycles of a function.

Ideas to be aware of

- ◇ A fixed point of a function f is also a fixed point of f^2; fixed points of f^2 that are not fixed points of f occur in pairs that form 2-cycles.

Solutions to Activities

Solution 1.1

(a) The function being iterated is

$$f(x) = x^2 + 1.$$

(b) The function being iterated is

$$f(x) = \frac{1}{2}\left(x + \frac{3}{x}\right).$$

Solution 1.2

(a) The first five terms are

0, 0, 0, 0, 0.

In this case, the sequence is constant.

(b) The first five terms are

1, 1, 1, 1, 1.

Once again, the sequence is constant.

(c) The first five terms are

$0.5,\ 0.25,\ 0.0625,\ 3.91 \times 10^{-3},\ 1.53 \times 10^{-5}.$

In this case, the sequence appears to tend (rapidly) to 0.

(d) The first five terms are

1.2, 1.44, 2.07, 4.30, 18.5.

In this case, the sequence appears to tend to infinity.

(e) The first five terms are

0.9, 0.81, 0.656, 0.430, 0.185.

In this case, the sequence appears to tend to 0.

(f) The first five terms are

-0.9, 0.81, 0.656, 0.430, 0.185.

In this case, the sequence appears to tend to 0. (Apart from the first term, all the terms are the same as those in part (e).)

Solution 1.3

In each case, the diagram confirms the long-term behaviour conjectured in Solution 1.2.

(a)

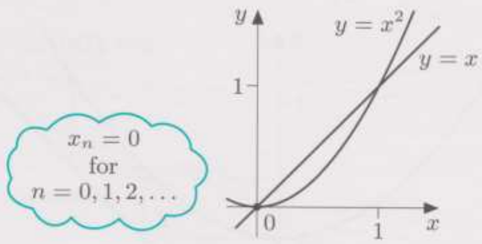


Figure S.1 x_n constant

(b)

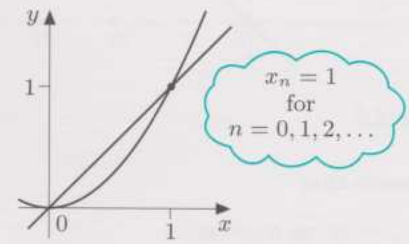


Figure S.2 x_n constant

(c)

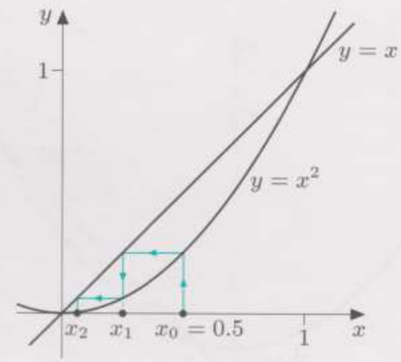


Figure S.3 $x_n \to 0$ as $n \to \infty$

Without drawing a very large graph, it is not possible to perform many stages of this construction.

(d)

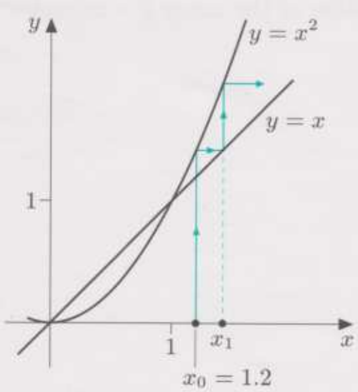


Figure S.4 $x_n \to \infty$ as $n \to \infty$

Solution 1.4

(a) $f(x) = x^2 + \frac{1}{2}$

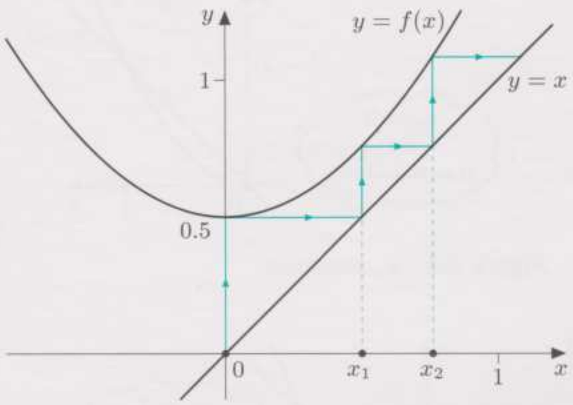


Figure S.5

It appears that

$$x_n \to \infty \text{ as } n \to \infty.$$

(b) $f(x) = x^2 + \frac{1}{8}$

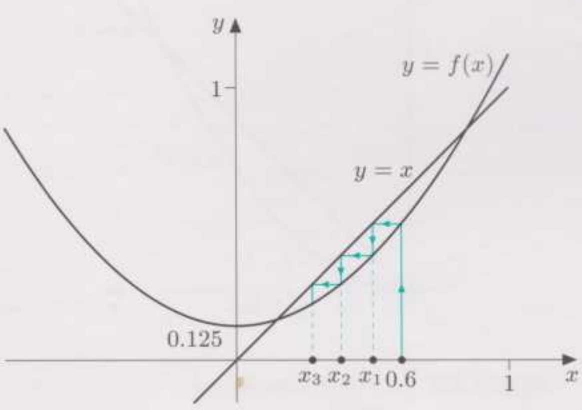


Figure S.6

It appears that the sequence x_n tends to a, where the point (a, a) is the left-hand point of intersection of the curve $y = x^2 + \frac{1}{8}$ and the line $y = x$.

(c) $f(x) = x^2 - \frac{1}{2}$

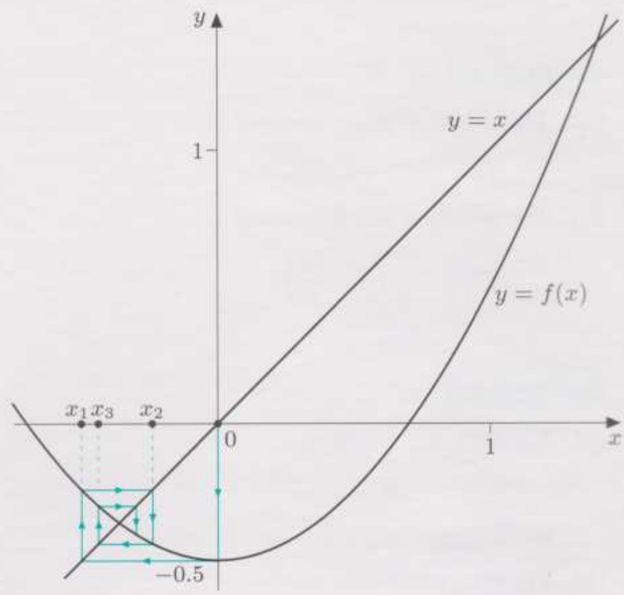


Figure S.7

It appears that the sequence x_n tends to a, where the point (a, a) is the left-hand point of intersection of the curve $y = x^2 - \frac{1}{2}$ and the line $y = x$.

(d) $f(x) = x^2 - 1$

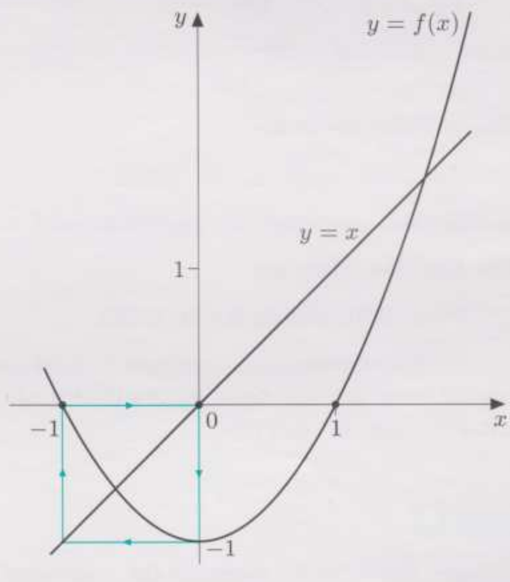


Figure S.8

The sequence x_n repeats the values

$$0,\ -1,\ 0,\ -1,\ 0,\ -1,\ \ldots.$$

Solution 1.5

For the function $f(x) = x^2 - \frac{1}{2}$, the fixed point equation is

$$x^2 - \tfrac{1}{2} = x; \quad \text{that is,} \quad x^2 - x - \tfrac{1}{2} = 0,$$

which has solutions $\frac{1}{2} \pm \frac{1}{2}\sqrt{3}$. Of these fixed points, only

$$\tfrac{1}{2} - \tfrac{1}{2}\sqrt{3} \simeq -0.366$$

lies between -0.5 and 0, so it is the left-hand fixed point in Figure S.7. Thus, by the Fixed Point Rule, this iteration sequence x_n converges to approximately -0.366.

Solution 1.6

(a) The fixed point equation is

$$-\tfrac{1}{8}x^2 + \tfrac{11}{8}x + \tfrac{1}{2} = x; \quad \text{that is,} \quad x^2 - 3x - 4 = 0,$$

which has solutions 4 and -1. These are indicated in Figure S.9.

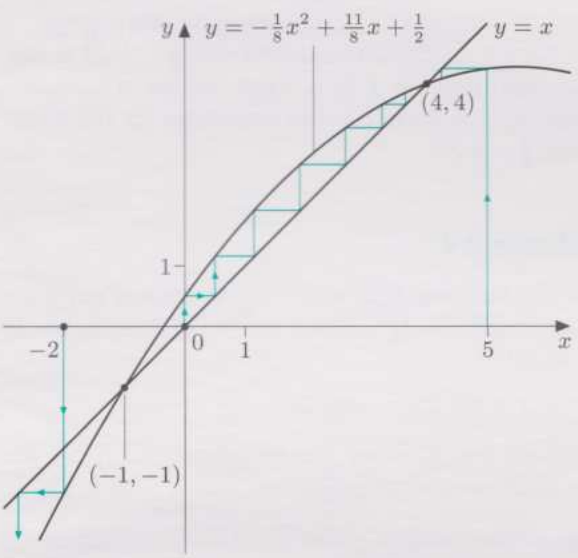


Figure S.9

(b) The effect of graphical iteration with the three initial terms $x_0 = 0$, $x_0 = -2$ and $x_0 = 5$ is also indicated in Figure S.9.

(i) If $x_0 = 0$, then $x_n \to 4$ as $n \to \infty$.

(ii) If $x_0 = -2$, then $x_n \to -\infty$ as $n \to \infty$.

(iii) If $x_0 = 5$, then $x_n \to 4$ as $n \to \infty$.

Solution 2.1

(a) If $f(x) = x^2$, then $f'(x) = 2x$.
Thus the gradient at the point $(\frac{1}{2}, \frac{1}{4})$ is

$$f'(\tfrac{1}{2}) = 2 \times (\tfrac{1}{2}) = 1.$$

(b) If $f(x) = -x^2 + 1$, then $f'(x) = -2x$.
Thus the gradient at the point $(3, -8)$ is

$$f'(3) = -2 \times 3 = -6.$$

(c) If $f(x) = x^2 - 2$, then $f'(x) = 2x$.
Thus the gradient at the point $(-\sqrt{2}, 0)$ is

$$f'(-\sqrt{2}\,) = 2 \times (-\sqrt{2}\,) = -2\sqrt{2}.$$

(d) If $f(x) = \frac{1}{3}x^2 + 7x + 1$, then $f'(x) = \frac{2}{3}x + 7$.
Thus the gradient at the point $(2, \frac{49}{3})$ is

$$f'(2) = \tfrac{2}{3} \times 2 + 7 = \tfrac{25}{3}.$$

Solution 2.2

(a) For $f(x) = x^2 + \frac{1}{2}$, the fixed point equation is

$$x^2 + \tfrac{1}{2} = x; \quad \text{that is,} \quad x^2 - x + \tfrac{1}{2} = 0,$$

which has no solutions since

$$(-1)^2 - 4 \times 1 \times \tfrac{1}{2} = -1 < 0.$$

So there are no fixed points to classify, as is clear from Figure S.5.

(b) For $f(x) = x^2 + \frac{1}{8}$, we know from the discussion before Activity 1.5 that the fixed points are $\frac{1}{2} \pm \frac{1}{4}\sqrt{2}$.

Now $f'(x) = 2x$, and hence

$$\begin{aligned} f'(\tfrac{1}{2} - \tfrac{1}{4}\sqrt{2}\,) &= 2 \times (\tfrac{1}{2} - \tfrac{1}{4}\sqrt{2}\,) \\ &= 1 - \tfrac{1}{2}\sqrt{2} \\ &\simeq 0.293 < 1, \\ f'(\tfrac{1}{2} + \tfrac{1}{4}\sqrt{2}\,) &= 2 \times (\tfrac{1}{2} + \tfrac{1}{4}\sqrt{2}\,) \\ &= 1 + \tfrac{1}{2}\sqrt{2} \\ &\simeq 1.707 > 1. \end{aligned}$$

Thus $\frac{1}{2} - \frac{1}{4}\sqrt{2}$ is an attracting fixed point of the function $f(x) = x^2 + \frac{1}{8}$, whereas $\frac{1}{2} + \frac{1}{4}\sqrt{2}$ is a repelling fixed point. These properties can be seen in Figure S.6, where the iteration sequence starting at 0.6 converges to the attracting fixed point.

(c) For $f(x) = x^2 - \frac{1}{2}$, we know from the solution to Activity 1.5 that the fixed points are $\frac{1}{2} \pm \frac{1}{2}\sqrt{3}$.

Now $f'(x) = 2x$, and hence

$$\begin{aligned} f'(\tfrac{1}{2} - \tfrac{1}{2}\sqrt{3}) &= 2 \times (\tfrac{1}{2} - \tfrac{1}{2}\sqrt{3}) \\ &= 1 - \sqrt{3} \\ &\simeq -0.732 > -1, \\ f'(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{3}) &= 2 \times (\tfrac{1}{2} + \tfrac{1}{2}\sqrt{3}) \\ &= 1 + \sqrt{3} \\ &\simeq 2.732 > 1. \end{aligned}$$

Thus $\frac{1}{2} - \frac{1}{2}\sqrt{3}$ is an attracting fixed point of the function $f(x) = x^2 - \frac{1}{2}$, whereas $\frac{1}{2} + \frac{1}{2}\sqrt{3}$ is a repelling fixed point. These properties can be seen in Figure S.7, where the iteration sequence starting at 0 converges to the attracting fixed point.

(d) For $f(x) = x^2 - 1$, the fixed point equation is

$$x^2 - 1 = x; \quad \text{that is,} \quad x^2 - x - 1 = 0,$$

which has solutions $x = \frac{1}{2} \pm \frac{1}{2}\sqrt{5}$.

Now $f'(x) = 2x$, and hence

$$\begin{aligned} f'(\tfrac{1}{2} - \tfrac{1}{2}\sqrt{5}) &= 2 \times (\tfrac{1}{2} - \tfrac{1}{2}\sqrt{5}) \\ &= 1 - \sqrt{5} \\ &\simeq -1.236 < -1, \\ f'(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{5}) &= 2 \times (\tfrac{1}{2} + \tfrac{1}{2}\sqrt{5}) \\ &= 1 + \sqrt{5} \\ &\simeq 3.236 > 1. \end{aligned}$$

Thus both fixed points are repelling. These properties can be seen in Figure S.8, though the iteration sequence x_n is not helpful in this case.

(e) For $f(x) = -\frac{1}{8}x^2 + \frac{11}{8}x + \frac{1}{2}$, we know from the solution to Activity 1.6 that the fixed points are -1 and 4.

Now $f'(x) = -\frac{1}{4}x + \frac{11}{8}$, and hence

$$\begin{aligned} f'(-1) &= -\tfrac{1}{4} \times (-1) + \tfrac{11}{8} = \tfrac{13}{8} > 1, \\ f'(4) &= -\tfrac{1}{4} \times 4 + \tfrac{11}{8} = \tfrac{3}{8} < 1. \end{aligned}$$

Thus -1 is a repelling fixed point of the function $f(x) = -\frac{1}{8}x^2 + \frac{11}{8}x + \frac{1}{2}$, whereas 4 is an attracting fixed point. These properties can be seen in Figure S.9, where the iteration sequences starting at 0 and 5 converge to the attracting fixed point 4, and the iteration sequences starting at 0 and -2 move away from the repelling fixed point -1.

Solution 2.3

Using the solutions to Activities 1.4(b) and 2.2(b), we obtain Figure S.10.

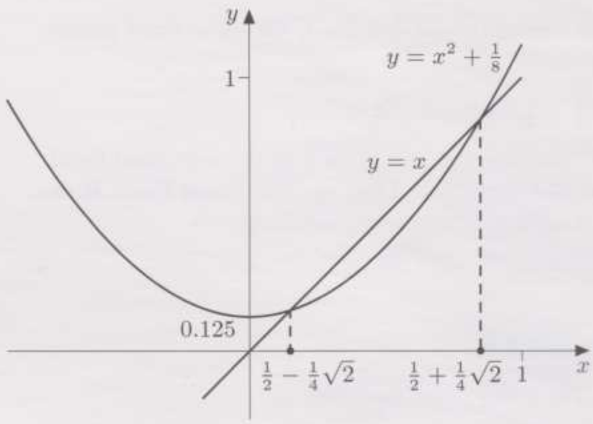


Figure S.10

First, we note that $\frac{1}{2} - \frac{1}{4}\sqrt{2}$ is an attracting fixed point of f, by the solution to Activity 2.2(b).

Next, the function f is increasing on the interval $I = (0, \frac{1}{2} + \frac{1}{4}\sqrt{2})$, for example, and $\frac{1}{2} - \frac{1}{4}\sqrt{2}$ is the only fixed point of f in I. Thus, by the graphical criterion, I is an interval of attraction for the fixed point $\frac{1}{2} - \frac{1}{4}\sqrt{2}$.

Solution 2.4

For the function $f(x) = x^2 - \frac{1}{2}$, we have $f'(x) = 2x$, so the condition $|f'(x)| < 1$ can be written as the two inequalities

$$-1 < 2x < 1; \quad \text{that is,} \quad -\tfrac{1}{2} < x < \tfrac{1}{2}.$$

Thus the set of values of x for which $|f'(x)| < 1$ is the open interval $(-\frac{1}{2}, \frac{1}{2})$.

The attracting fixed point $\frac{1}{2} - \frac{1}{2}\sqrt{3} \simeq -0.366$ is nearer to $-\frac{1}{2}$ than to $\frac{1}{2}$, so, in order to satisfy the gradient criterion, we choose an interval of attraction I to have left-hand endpoint $-\frac{1}{2}$. In order that $\frac{1}{2} - \frac{1}{2}\sqrt{3}$ is the midpoint of I, we take the right-hand endpoint to be

$$\begin{aligned} \tfrac{1}{2} - \tfrac{1}{2}\sqrt{3} + \left((\tfrac{1}{2} - \tfrac{1}{2}\sqrt{3}) - (-\tfrac{1}{2})\right) &= \tfrac{3}{2} - \sqrt{3} \\ &\simeq -0.232. \end{aligned}$$

Thus an interval of attraction for $\frac{1}{2} - \frac{1}{2}\sqrt{3}$ is

$$I = (-\tfrac{1}{2}, \tfrac{3}{2} - \sqrt{3}).$$

Solution 3.1

(a) The domain of f is $A = (-\frac{1}{2}\pi, \frac{1}{2}\pi)$, and its codomain is $B = \mathbb{R}$. The graph of $y = f(x)$ is shown in Figure S.11.

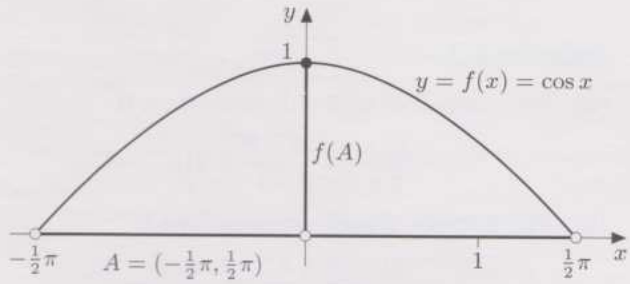


Figure S.11

The image set of f is $f(A) = (0, 1]$.

The domain of g is $C = (0, \infty)$, and its codomain is $D = \mathbb{R}$. Since $(0, 1] \subseteq (0, \infty)$, we have $f(A) \subseteq C$, so the composite function $g \circ f$ exists and is given by

$$g \circ f : (-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi) \longrightarrow \mathbb{R}$$
$$x \longmapsto g(f(x)) = \frac{1}{\cos x} = \sec x.$$

(b) The domain of g is $C = (0, \infty)$, and its codomain is $D = \mathbb{R}$. The graph of $y = g(x)$ is shown in Figure S.12.

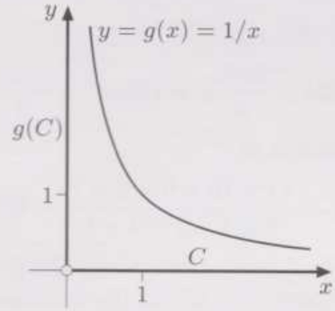


Figure S.12

The image set of g is $g(C) = (0, \infty)$.

The domain of h is $E = [0, \infty)$, and its codomain is $F = \mathbb{R}$. Since $(0, \infty) \subseteq [0, \infty)$, we have $g(C) \subseteq E$, so the composite function $h \circ g$ exists and is given by

$$h \circ g : (0, \infty) \longrightarrow \mathbb{R}$$
$$x \longmapsto h(g(x)) = \sqrt{\frac{1}{x}} = \frac{1}{\sqrt{x}}.$$

Solution 3.2

(a) We check first that $f(a) = b$ and then that $f(b) = a$:

$$f(-\tfrac{1}{4}) = (-\tfrac{1}{4})^2 - \tfrac{13}{16} = -\tfrac{3}{4},$$
$$f(-\tfrac{3}{4}) = (-\tfrac{3}{4})^2 - \tfrac{13}{16} = -\tfrac{1}{4}.$$

Thus $a = -\frac{1}{4}$, $b = -\frac{3}{4}$ form a 2-cycle of the function $f(x) = x^2 - \frac{13}{16}$.

(b) In this case,

$$\begin{aligned} f(-\tfrac{1}{2} + \tfrac{1}{2}\sqrt{5}) &= (-\tfrac{1}{2} + \tfrac{1}{2}\sqrt{5})^2 - 2 \\ &= \tfrac{1}{4} - \tfrac{1}{2}\sqrt{5} + \tfrac{5}{4} - 2 \\ &= -\tfrac{1}{2} - \tfrac{1}{2}\sqrt{5}, \\ f(-\tfrac{1}{2} - \tfrac{1}{2}\sqrt{5}) &= (-\tfrac{1}{2} - \tfrac{1}{2}\sqrt{5})^2 - 2 \\ &= \tfrac{1}{4} + \tfrac{1}{2}\sqrt{5} + \tfrac{5}{4} - 2 \\ &= -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{5}. \end{aligned}$$

Thus $a = -\frac{1}{2} + \frac{1}{2}\sqrt{5}$, $b = -\frac{1}{2} - \frac{1}{2}\sqrt{5}$ form a 2-cycle of the function $f(x) = x^2 - 2$.

(In decimals, the calculation showing that $f(a) = b$ would be as follows:

$$\begin{aligned} f(a) &= f(-\tfrac{1}{2} + \tfrac{1}{2}\sqrt{5}) \\ &= (-\tfrac{1}{2} + \tfrac{1}{2}\sqrt{5})^2 - 2 \\ &= -1.618\,033\,98 \end{aligned}$$

and

$$b = -\tfrac{1}{2} - \tfrac{1}{2}\sqrt{5} = -1.618\,033\,98,$$

as required. For each calculation, a full calculator display is given.)

Solution 3.3

(a) For $f(x) = x^2 - \frac{13}{16}$, we have $f'(x) = 2x$, so

$$\begin{aligned} f'(a)f'(b) &= (2a) \times (2b) \\ &= (-\tfrac{1}{2}) \times (-\tfrac{3}{2}) \\ &= \tfrac{3}{4}. \end{aligned}$$

Since $|\frac{3}{4}| < 1$, the 2-cycle a, b is attracting.

The first six terms are

$$\begin{aligned} &x_0 = 0, & &x_1 = -0.813, \\ &x_2 = -0.152, & &x_3 = -0.789, \\ &x_4 = -0.190, & &x_5 = -0.777. \end{aligned}$$

Thus it seems possible that x_n tends to the 2-cycle $a = -0.25$, $b = -0.75$. (Further calculation shows that this does occur.)

(b) For $f(x) = x^2 - 2$, we have $f'(x) = 2x$, so

$$\begin{aligned} f'(a)f'(b) &= (2a) \times (2b) \\ &= (-1+\sqrt{5})(-1-\sqrt{5}) \\ &= -4. \end{aligned}$$

Since $|-4| > 1$, the 2-cycle a, b is repelling.

The first six terms are

$$\begin{aligned} &x_0 = 0.5, &&x_1 = -1.75, \\ &x_2 = 1.06, &&x_3 = -0.871, \\ &x_4 = -1.24, &&x_5 = -0.459. \end{aligned}$$

Thus x_n does not seem to tend to the 2-cycle $a \simeq 1.618$, $b \simeq -1.618$.

Solution 5.1

(a) 1, 6, 15, 20, 15, 6, 1;
1, 7, 21, 35, 35, 21, 7, 1.

(b) (i) $$\begin{aligned}(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 \\ + 15a^2b^4 + 6ab^5 + b^6\end{aligned}$$

(ii) $$\begin{aligned}(1+x)^7 = 1 + 7x + 21x^2 + 35x^3 + 35x^4 \\ + 21x^5 + 7x^6 + x^7\end{aligned}$$

(iii) $$\begin{aligned}(1+2x)^6 &= 1 + 6(2x) + 15(2x)^2 + 20(2x)^3 \\ &\quad + 15(2x)^4 + 6(2x)^5 + (2x)^6 \\ &= 1 + 12x + 60x^2 + 160x^3 + 240x^4 \\ &\quad + 192x^5 + 64x^6\end{aligned}$$

(c) Using the sixth row of Pascal's triangle, we obtain

$$\begin{aligned} &(2 - \tfrac{1}{2}x)^5 \\ &= 2^5 + 5 \times 2^4(-\tfrac{1}{2}x) + 10 \times 2^3(-\tfrac{1}{2}x)^2 \\ &\quad + 10 \times 2^2(-\tfrac{1}{2}x)^3 + 5 \times 2(-\tfrac{1}{2}x)^4 + (-\tfrac{1}{2}x)^5 \\ &= 32 - 40x + 20x^2 - 5x^3 + \tfrac{5}{8}x^4 - \tfrac{1}{32}x^5. \end{aligned}$$

Solution 5.2

(a) There are $4! = 24$ permutations of A, B, C, D.

(b)

ABCD	BACD	CABD	DABC
ABDC	BADC	CADB	DACB
ACBD	BCAD	CBAD	DBAC
ACDB	BCDA	CBDA	DBCA
ADBC	BDAC	CDAB	DCAB
ADCB	BDCA	CDBA	DCBA

Solution 5.3

There are

$${}^8P_5 = 8 \times 7 \times 6 \times 5 \times 4 = 6720$$

such permutations.

Solution 5.4

(a) The number of such subcommittees is

$${}^{10}C_4 = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = 210.$$

(b) The number of such combinations is

$$\begin{aligned} {}^{49}C_6 &= \frac{49 \times 48 \times 47 \times 46 \times 45 \times 44}{6 \times 5 \times 4 \times 3 \times 2 \times 1} \\ &= 13\,983\,816. \end{aligned}$$

Solution 5.5

(a) (i) The first four terms of $(1+x)^{10}$ are

$$\begin{aligned} &1 + {}^{10}C_1 x + {}^{10}C_2 x^2 + {}^{10}C_3 x^3 \\ &= 1 + 10x + \left(\frac{10 \times 9}{2 \times 1}\right)x^2 + \left(\frac{10 \times 9 \times 8}{3 \times 2 \times 1}\right)x^3 \\ &= 1 + 10x + 45x^2 + 120x^3. \end{aligned}$$

(ii) Using the binomial coefficients obtained in part (a)(i), we find that the first four terms of $(2 + \frac{1}{3}x)^{10}$ are

$$\begin{aligned} &2^{10} + 10 \times 2^9 \left(\tfrac{1}{3}x\right) + 45 \times 2^8 \left(\tfrac{1}{3}x\right)^2 \\ &\quad + 120 \times 2^7 \left(\tfrac{1}{3}x\right)^3 \\ &= 1024 + \frac{5120}{3}x + 1280x^2 + \frac{5120}{9}x^3. \end{aligned}$$

(b) The coefficient is

$${}^{11}C_5 = \frac{11 \times 10 \times 9 \times 8 \times 7}{5 \times 4 \times 3 \times 2 \times 1} = 462.$$

(c) The constant term arises when the power of x and the power of $-1/x$ in the expansion are the same, namely 6. The term in $x^6(-1/x)^6$ is constant and is given by

$${}^{12}C_6 = \frac{12 \times 11 \times 10 \times 9 \times 8 \times 7}{6 \times 5 \times 4 \times 3 \times 2 \times 1} = 924.$$

Solutions to Exercises

Solution 1.1

(a) The first four terms are (to 3 s.f.):

0.5, 0.25, 0.188, 0.152.

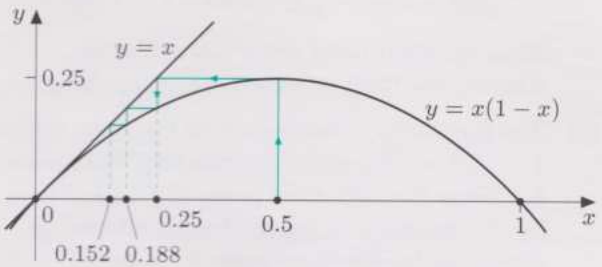


Figure S.13

(b) The first four terms are (to 3 s.f.):

1, 2, 1.75, 1.73.

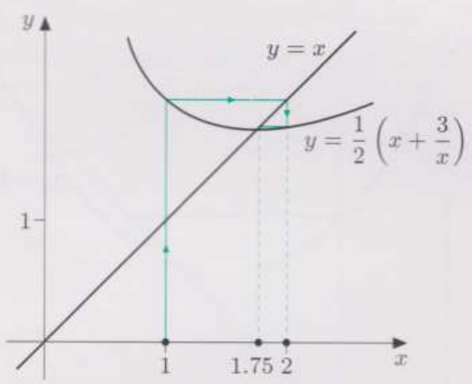


Figure S.14

(c) The first four terms are (to 3 s.f.):

0, 1, 0.54, 0.858.

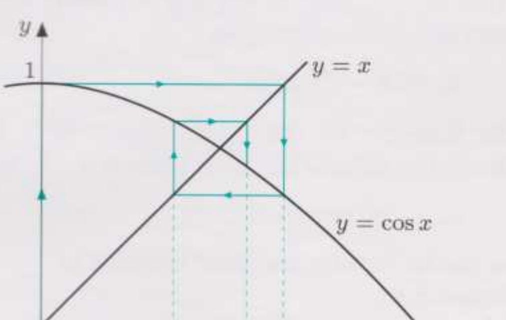


Figure S.15

(d) The first four terms are (to 3 s.f.):

0, −2.4, 3.36, 8.89.

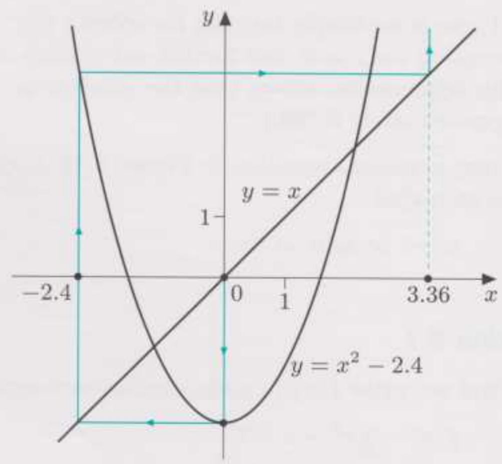


Figure S.16

Solution 1.2

(a) By the Fixed Point Rule, the limit is a fixed point of the function $f(x) = x(1 - x)$. The fixed point equation is

$$x(1 - x) = x,$$

which can be rearranged as

$$x^2 = 0.$$

Thus the only fixed point is 0, so this must be the limit of the sequence.

(b) By the Fixed Point Rule, the limit is a fixed point of the function $f(x) = \frac{1}{2}(x + 3/x)$. The fixed point equation is

$$\tfrac{1}{2}(x + 3/x) = x,$$

which can be rearranged as

$$3/x = x; \quad \text{that is,} \quad x^2 = 3.$$

Thus the fixed points are $\pm\sqrt{3}$.

We also know, from Figure S.14, that the limit lies between 1 and 2, so it must be $\sqrt{3}$.

(This iteration sequence arises as a special case of a general method of solving equations, called the Newton–Raphson method, which you will meet in Block C. Here the method has been applied to the equation $x^2 - 3 = 0$.)

Solution 1.3

(a) By the Fixed Point Rule, the limit is a fixed point of the function $f(x) = \cos x$; that is, it is a solution of the equation

$$\cos x = x.$$

Also, from Figure S.15, the limit lies between 0.54 and 0.858.

(There is no simple formula for solving the equation $\cos x = x$, but further calculation of this sequence x_n shows that the solution is approximately 0.739.)

(b) From graphical iteration in Figure S.16, it can be seen that

$$x_n \to \infty \text{ as } n \to \infty.$$

Solution 2.1

(a) First we write $f(x)$ in completed-square form:

$$\begin{aligned} f(x) &= \tfrac{1}{8}x^2 - x + 7 \\ &= \tfrac{1}{8}(x^2 - 8x) + 7 \\ &= \tfrac{1}{8}\left((x-4)^2 - 16\right) + 7 \\ &= \tfrac{1}{8}(x-4)^2 + 5. \end{aligned}$$

Hence the vertex of the parabola has coordinates $(4, 5)$, and it is the lowest point of the graph of f.

The y-intercept is $f(0) = 7$, and there are no x-intercepts since the equation $f(x) = 0$ has no real solutions. Thus the graph of $y = f(x)$ is as shown in Figure S.17.

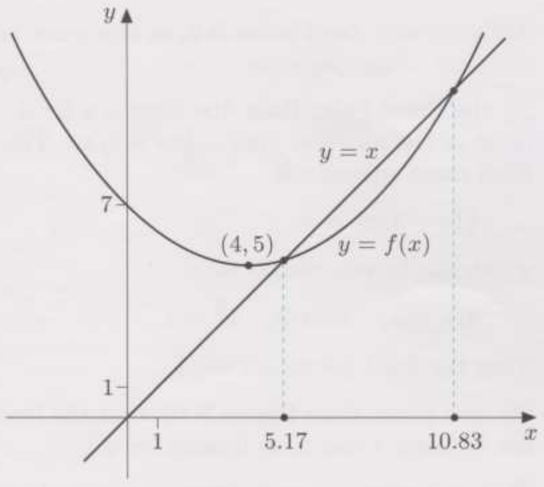


Figure S.17

(b) The fixed point equation is

$$\tfrac{1}{8}x^2 - x + 7 = x; \quad \text{that is,} \quad x^2 - 16x + 56 = 0,$$

which has solutions $8 + \sqrt{8} \simeq 10.83$ and $8 - \sqrt{8} \simeq 5.17$. These are indicated in Figure S.17.

Now $f'(x) = \frac{1}{4}x - 1$, and hence

$$f'(8 - \sqrt{8}) = \tfrac{1}{4}(8 - \sqrt{8}) - 1 \simeq 0.293 < 1,$$
$$f'(8 + \sqrt{8}) = \tfrac{1}{4}(8 + \sqrt{8}) - 1 \simeq 1.707 > 1.$$

Thus the fixed point $8 - \sqrt{8}$ is attracting, whereas the fixed point $8 + \sqrt{8}$ is repelling.

(c) The function f is increasing on the open interval $I = (4, 8 + \sqrt{8})$, and the attracting fixed point $8 - \sqrt{8}$ is the only fixed point of f in I. Thus, by the graphical criterion, I is an interval of attraction for the fixed point $8 - \sqrt{8}$.

(d) The effect of graphical iteration with the initial terms $x_0 = 0$ and $x_0 = -3.5$ is shown in Figure S.18.

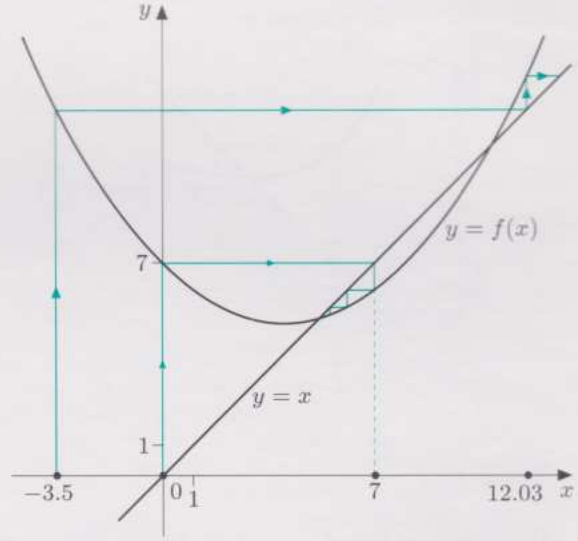


Figure S.18

(i) If $x_0 = 0$, then $x_1 = f(x_0) = 7$ lies in the interval of attraction I, so

$$x_n \to 8 - \sqrt{8} \text{ as } n \to \infty.$$

(ii) If $x_0 = -3.5$, then $x_1 = f(x_0) = \frac{385}{32} \simeq 12.03$ lies to the right of the fixed point $8 + \sqrt{8}$, so

$$x_n \to \infty \text{ as } n \to \infty,$$

as can be seen by graphical iteration in Figure S.18.

Solution 2.2

(a) By performing a y-scaling with factor 2.5 on the graph of $y = x(1-x)$, we find that the graph of f is as shown in Figure S.19.

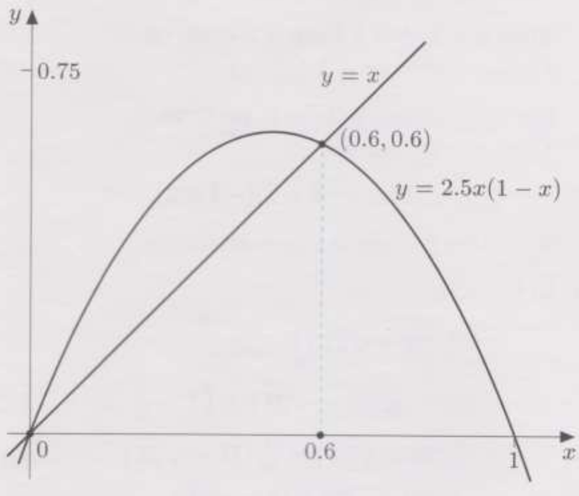


Figure S.19

(b) The fixed point equation is

$$2.5x(1-x) = x,$$

which can be rearranged as

$$1.5x - 2.5x^2 = 0; \quad \text{that is,} \quad 1.5x(1 - \tfrac{5}{3}x) = 0.$$

Thus the fixed points are 0 and $\frac{3}{5} = 0.6$. These are indicated in Figure S.19.

Now, since $f(x) = 2.5x - 2.5x^2$, $f'(x) = 2.5 - 5x$; hence

$$f'(0) = 2.5 > 1,$$
$$f'(0.6) = 2.5 - 5 \times 0.6 = -0.5 > -1.$$

Thus the fixed point 0.6 is attracting, whereas the fixed point 0 is repelling.

(c) By part (b), we know that 0.6 is an attracting fixed point of f.

Since $f'(x) = 2.5 - 5x$, the condition $|f'(x)| < 1$ can be written as the two inequalities

$$-1 < 2.5 - 5x < 1.$$

We solve these inequalities in turn.

◇ $-1 < 2.5 - 5x$ is equivalent to

$$5x < 3.5; \quad \text{that is,} \quad x < \frac{3.5}{5} = 0.7.$$

◇ $2.5 - 5x < 1$ is equivalent to

$$1.5 < 5x; \quad \text{that is,} \quad x > \frac{1.5}{5} = 0.3.$$

Thus the condition $|f'(x)| < 1$ is equivalent to

$$0.3 < x < 0.7.$$

Hence the set of values of x for which $|f'(x)| < 1$ is the open interval $(0.3, 0.7)$.

The attracting fixed point 0.6 is nearer to 0.7 than to 0.3, so we choose an interval of attraction I to have right-hand endpoint 0.7. In order that 0.6 is the midpoint of I, we take the left-hand endpoint to be 0.5. Thus an interval of attraction for 0.6 is $I = (0.5, 0.7)$.

(d) The effect of graphical iteration with the initial terms $x_0 = 0$ and $x_0 = 0.5$ is shown in Figure S.20.

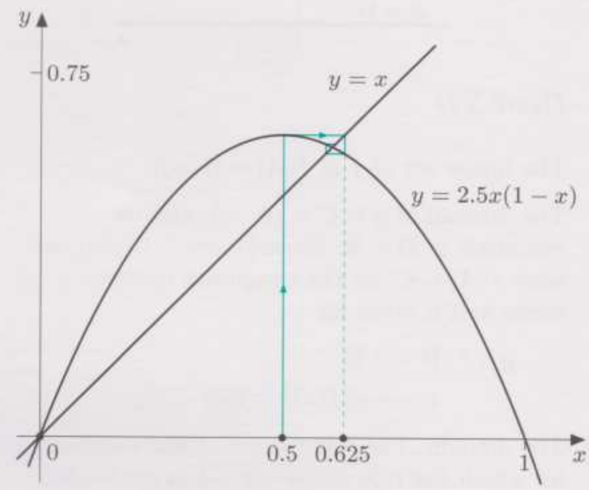


Figure S.20

(i) If $x_0 = 0$, then $x_n = 0$ for all n, since 0 is a fixed point, so

$$x_n \to 0 \text{ as } n \to \infty.$$

(ii) If $x_0 = 0.5$, then $x_1 = f(x_0) = 0.625$ lies in the interval of attraction I, so

$$x_n \to 0.6 \text{ as } n \to \infty.$$

(Notice that the iteration sequence in Figure 1.2(a) is generated by this function f, and that the sequence in Figure 1.2(a) does indeed appear to tend to the attracting fixed point 0.6 identified here.)

Solution 3.1

(a) The domain of f is $A = \mathbb{R}$, and its codomain is $B = \mathbb{R}$. The graph of $y = f(x)$ is as shown in Figure S.21.

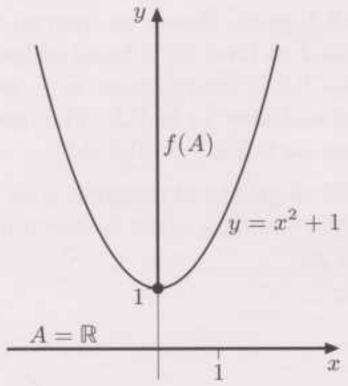


Figure S.21

The image set of f is $f(A) = [1, \infty)$.

The domain of g is $C = (0, \infty)$, and its codomain is $D = \mathbb{R}$. Since $[1, \infty) \subseteq (0, \infty)$, we have $f(A) \subseteq C$, so the composite function $g \circ f$ exists and is given by

$$\begin{aligned} g \circ f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto g(f(x)) = \ln(1 + x^2). \end{aligned}$$

(b) The domain A of f is the set of real numbers x for which the rule $x \longmapsto \sqrt{x+1}$ is applicable; that is, $A = [-1, \infty)$. The codomain of f is $B = \mathbb{R}$. The graph of $y = f(x)$ is as shown in Figure S.22.

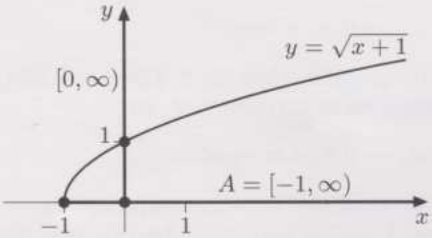


Figure S.22

The image set of f is $f(A) = [0, \infty)$.

The domain of g is $C = [-1, \infty)$, and its codomain is $D = \mathbb{R}$. Since $[0, \infty) \subseteq [-1, \infty)$, we have $f(A) \subseteq C$, so the composite function $g \circ f$ exists and is given by

$$\begin{aligned} g \circ f : [-1, \infty) &\longrightarrow \mathbb{R} \\ x &\longmapsto g(f(x)) = \sqrt{\sqrt{x+1} + 1}. \end{aligned}$$

Solution 3.2

(a) We check first that $f(a) = b$ and then that $f(b) = a$:

$$\begin{aligned} f(1) &= -1 + 2 + 1 = 2, \\ f(2) &= -4 + 4 + 1 = 1. \end{aligned}$$

Thus $a = 1$, $b = 2$ form a 2-cycle of $f(x) = -x^2 + 2x + 1$.

For $f(x) = -x^2 + 2x + 1$, we have $f'(x) = -2x + 2$, so

$$f'(a)f'(b) = (-2+2)(-4+2) = 0.$$

Thus this 2-cycle is super-attracting.

(b) In this case,

$$\begin{aligned} &f\left(\tfrac{1}{32}(21 + \sqrt{21}\,)\right) \\ &= 3.2 \times \tfrac{1}{32}(21 + \sqrt{21}\,) \times \left(1 - \tfrac{1}{32}(21 + \sqrt{21}\,)\right) \\ &= \tfrac{1}{10}(21 + \sqrt{21}\,) \times \tfrac{1}{32}(11 - \sqrt{21}\,) \\ &= \tfrac{1}{320}(21 + \sqrt{21}\,)(11 - \sqrt{21}\,) \\ &= \tfrac{1}{320}(210 - 10\sqrt{21}\,) \\ &= \tfrac{1}{32}(21 - \sqrt{21}\,) \end{aligned}$$

and

$$\begin{aligned} &f\left(\tfrac{1}{32}(21 - \sqrt{21}\,)\right) \\ &= 3.2 \times \tfrac{1}{32}(21 - \sqrt{21}\,) \times \left(1 - \tfrac{1}{32}(21 - \sqrt{21}\,)\right) \\ &= \tfrac{1}{10}(21 - \sqrt{21}\,) \times \tfrac{1}{32}(11 + \sqrt{21}\,) \\ &= \tfrac{1}{320}(21 - \sqrt{21}\,)(11 + \sqrt{21}\,) \\ &= \tfrac{1}{32}(21 + \sqrt{21}\,). \end{aligned}$$

Thus $a = \frac{1}{32}(21 + \sqrt{21}\,)$, $b = \frac{1}{32}(21 - \sqrt{21}\,)$ form a 2-cycle of $f(x) = 3.2x(1-x)$.

For $f(x) = 3.2x(1-x) = 3.2x - 3.2x^2$, we have $f'(x) = 3.2 - 6.4x$, so

$$\begin{aligned} f'(a)f'(b) &= \left(3.2 - 6.4 \times \tfrac{1}{32}(21 + \sqrt{21}\,)\right) \\ &\quad \times \left(3.2 - 6.4 \times \tfrac{1}{32}(21 - \sqrt{21}\,)\right) \\ &= \left(\frac{-5 - \sqrt{21}}{5}\right)\left(\frac{-5 + \sqrt{21}}{5}\right) \\ &= \frac{4}{25} < 1. \end{aligned}$$

Thus this 2-cycle is attracting.

(Notice that the iteration sequence in Figure 1.2(b) is generated by this function f, and that the sequence in Figure 1.2(b) does indeed appear to tend to the attracting 2-cycle

$$\begin{aligned} a &= \tfrac{1}{32}(21 + \sqrt{21}\,) \simeq 0.799, \\ b &= \tfrac{1}{32}(21 - \sqrt{21}\,) \simeq 0.513, \end{aligned}$$

identified here.)

Solution 5.1

The eighth row of Pascal's triangle is

$$1,\ 7,\ 21,\ 35,\ 35,\ 21,\ 7,\ 1;$$

see the solution to Activity 5.1(a). Thus the ninth row is

$$1,\ 8,\ 28,\ 56,\ 70,\ 56,\ 28,\ 8,\ 1.$$

Using the above coefficients, we obtain

$$\begin{aligned}(1+x)^8 = 1 + 8x + 28x^2 + 56x^3 + 70x^4\\ + 56x^5 + 28x^6 + 8x^7 + x^8.\end{aligned}$$

Solution 5.2

(a) The number of six-letter permutations from A, B, C, D, E, F, G, H, I, J is

$$^{10}P_6 = 10 \times 9 \times 8 \times 7 \times 6 \times 5 = 151\,200.$$

(b) The number of six-letter combinations from A, B, C, D, E, F, G, H, I, J is

$$^{10}C_6 = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5}{6 \times 5 \times 4 \times 3 \times 2 \times 1} = 210.$$

Solution 5.3

(a) The x^6 term in the expansion of $(1+2x)^6$ is $(2x)^6$, so the required coefficient is $2^6 = 64$.

(b) The x^6 term in the expansion of $(2-3x)^{10}$ is

$$\begin{aligned}^{10}C_6 \times 2^4 \times (-3x)^6 &= 210 \times 16 \times 729x^6\\ &= 2\,449\,440x^6,\end{aligned}$$

using the solution to Exercise 5.2(b), so the required coefficient is 2 449 440.

(c) The x^6 term in the expansion of $(5-x^2)^5$ is

$$\begin{aligned}^5C_3 \times 5^2 \times (-x^2)^3 &= -\left(\frac{5 \times 4 \times 3}{3 \times 2 \times 1}\right) \times 25x^6\\ &= -250x^6,\end{aligned}$$

so the required coefficient is -250.

Solution 5.4

(a) The first three terms of $(a+b)^{12}$ are

$$\begin{aligned}&a^{12} + {}^{12}C_1 a^{11} b + {}^{12}C_2 a^{10} b^2\\ &= a^{12} + 12a^{11}b + \left(\frac{12 \times 11}{2 \times 1}\right) a^{10}b^2\\ &= a^{12} + 12a^{11}b + 66a^{10}b^2.\end{aligned}$$

(b) Using the binomial coefficients obtained in part (a), we find that the first three terms of $(3+\frac{1}{2}x)^{12}$ are

$$\begin{aligned}&3^{12} + 12 \times 3^{11} \times \left(\frac{x}{2}\right) + 66 \times 3^{10} \times \left(\frac{x}{2}\right)^2\\ &= 531\,441 + 1\,062\,882x + \left(\frac{1\,948\,617}{2}\right) x^2.\end{aligned}$$

Index

The Open University
ISBN 0 7492 3998 1